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Stochastic Adsorption-Desorption With a New Discontinuous Galerkin Method

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Abstract. *This paper is devoted to the study of a new discontinuous finite element idea of a nonuniform rational B-spline (NURBS) basis for solving a stochastic adsorption-desorption problem. In addition to existence and uniqueness of solution, we give a posteriori error and order of convergence for this scheme. Some verifying experimental results are also reported.*

Key words: Discontinuous Galerkin Method, Stochastic Adsorption-Desorption Problem, Nanotechnology, Mass-Spring Systems, Error Estimate, NURBS.

AMS Subject Classifications: 65N60, 65N15

1. Introduction

Because of their stability and high accuracy, discontinuous Galerkin (DG) methods can produce better results (see e.g. [2, 3, 4, 5, 6, 7, 10, 12]) than standard finite element methods for both smooth and non-smooth solutions in non-stochastic problems In such deterministic problems artificial diffusion is added only in the characteristic direction so that internal layers are not smeared out, while the added diffusion removes oscillations near boundary layers. Moreover, in the biotechnology for mass-spring systems physical phenomena can give rise to stochastic adsorption and desorption behavior. We observe, from the outset of this paper, that this stochastic subject can model such phenomena via introducing a concept of Poisson distributed stochastic process governed by a stochastic differential equation [16].

Let $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$ be a complete probability space with respect to the normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Also, assume that $\varphi : [0,+\infty) \to (0,+\infty)$ is a function in an L_2 valued Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$. Here, $(N(\varphi(t)))_{t\in[0,+\infty]}$ is a Poisson distributed stochastic process with parameter $\varphi(t)$ if for each $t \in [0,+\infty]$, $N(\varphi(t))$ is a random variable on $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$ that takes its value in $\mathbb{Z}_+$ and satisfies the distribution

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$$P(N(\varphi(t)) = k) = \frac{e^{-\varphi(t)}(\varphi(t))^k}{k!}, \ k \in \mathbb{Z}_+ . \tag{1}$$

The stochastic adsorption-desorption problem (see [11]) for mass-spring systems (or kinetic systems) is developed, when $x(0)$, $x'(0)$ or $\mathbf{E}\left(|x(0)|^2\right)$, $\mathbf{E}\left(|x'(0)|^2\right)$ are given , as follows:

$$\left.\begin{array}{ll} \theta\, x''(t) + c\, x(t) = \theta f(t,x(t)), & t \in [0,S] \\ \mathbf{E}\left(|x(0)|^2\right) = \eta_1 , & 0 \le \eta_1 \le \beta \\ \mathbf{E}\left(|x'(0)|^2\right) = \eta_2 , & 0 \le \eta_2 \le \alpha \end{array}\right\} . \tag{2}$$

If $\beta, \alpha < +\infty$ and $m_0,\, m_a,\, c \in \mathbb{R}_+ \backslash \{0\}$ and $\theta = m_0 + m_a \times N(\varphi(t))$, then we can show that (2) has a unique continuous solution in L_2 with $\mathbf{E}\left(\sup_{t \le S} |x(t)|^2\right) \le +\infty$. So let us introduce the variables $x_1(t) = x(t)$ and $x_2(t) = x'(t)$ to rewrite (2) as

$$\left.\begin{array}{ll} x_1'(t) - x_2(t) = 0, & t \in [0,S] \\ x_2'(t) + \dfrac{c}{m_0 + m_a \times N(\varphi(t))}\, x_1(t) = f(t,x(t)), & t \in [0,S] \\ \mathbf{E}\left(|x_1(0)|^2\right) = \eta_1 , & 0 \le \eta_1 \le \beta \\ \mathbf{E}\left(|x_2(0)|^2\right) = \eta_2 , & 0 \le \eta_2 \le \alpha \end{array}\right\} . \tag{3}$$

In system form, (3) is the same as

$$\left.\begin{array}{l} d\,\mathbf{u}(t) + A(t)\,\mathbf{u}(t) = F(t,x(t)), \ \ t \in [0,S] \\ \mathbf{u}(0) = \mathbf{u}_0, \end{array}\right\} . \tag{4}$$

where $\mathbf{u}(t) = (x_1(t),x_2(t))^T$, $d\mathbf{u}(t) = (dx_1(t),dx_2(t))^T$, $\mathbf{u}_0 = \mathbf{u}(0) = (x_1(0),x_2(0))^T$, (or $\mathbf{u}_0 = \left(\mathbf{E}\left(|x(0)|^2\right),\, \mathbf{E}\left(|x'(0)|^2\right)\right)^T$), $F(t,x(t)) = \left(0,\, f(t,x(t))\right)^T$ and

$$A(t) = \begin{pmatrix} 0 & -1 \\ \dfrac{c}{m_0 + m_a \times N(\varphi(t))} & 0 \end{pmatrix}.$$

The structure of the rest of this article is organized as follows. In Section 2, we present and analyze the DG-method for above stochastic adsorption-desorption problem. Existence and uniqueness of solution are investigated in section 3. In section 4, we explain the DG-method for this system. In section 5, we consider the dual problem and establish stability by applying a posteriori error estimates. We prove the order of convergence in section 6. Finally In section 7, we give and compare two algorithms with some numerical test results.

2. Existence and Uniqueness of Solution

In this section, we will present a proof for existence and uniqueness based on Poisson distributed stochastic process (also, we refer to [14]).

Theorem 2.1. *If $\beta, \alpha < +\infty$ and $m_0,\, m_a,\, c \in \mathbb{R}_+ \backslash \{0\}$, then (2) has a unique continuous solution in L_2 with*

$$\mathbf{E}\left(\sup_{t\le S}|x(t)|^2\right)\le+\infty\,.$$

Proof. If $x(t)$ is a strong solution of (2), then $\mathbf{u}(t)=(x(t),x'(t))^T$ is a strong solution of (4). This means that $\mathbf{u}$ is solution of the integral equation (see [13]):

$$\mathbf{u}(t)=\mathbf{u}_0-\int_0^t\sigma(t_1,\mathbf{u}(t_1))\,dP(t_1)+\int_0^t F(t_1,\mathbf{u}(t_1))dt_1\,,\quad t\in[0,S],\tag{5}$$

such that $\sigma(t,\mathbf{u}(t))dP(t)=d(A(t)\mathbf{u}(t))$and P is a Poisson distributed stochastic process. By using the inequality $(a+b+c)^2\le 3\,(a^2+b^2+c^2)$ we have:

$$\mathbf{E}\left(\sup_{t\le S}|\mathbf{u}(t)|^2\right)=\mathbf{E}\left(\sup_{t\le S}\left|\mathbf{u}_0-\int_0^t\sigma(t_1,\mathbf{u}(t_1))dP(t_1)+\int_0^t F(t_1,\mathbf{u}(t_1))dt_1\right|^2\right)$$
$$\le 3\left\{\mathbf{E}(|\mathbf{u}_0|^2)+\mathbf{E}\left(\sup_{t\le S}\left|\int_0^t(-\sigma(t_1,\mathbf{u}(t_1)))dP(t_1)\right|^2\right)+\mathbf{E}\left(\sup_{t\le S}\left|\int_0^t F(t_1,\mathbf{u}(t_1))dt_1\right|^2\right)\right\}.$$

We assume σ and F to be continuous and to satisfy a global Lipschitz condition,

$$\left|\sigma(t,\mathbf{u}(t))-\sigma(t,\hat{\mathbf{u}}(t))\right|\le L\left|\mathbf{u}(t)-\hat{\mathbf{u}}(t)\right|,\quad t\in[0,S],\tag{6}$$

$$\left|F(t,\mathbf{u}(t))-F(t,\hat{\mathbf{u}}(t))\right|\le L\left|\mathbf{u}(t)-\hat{\mathbf{u}}(t)\right|,\quad t\in[0,S],\tag{7}$$

and a linear growth bound

$$\left|\sigma(t,\mathbf{u}(t))\right|\le L\,(1+|\mathbf{u}(t)|),\quad t\in[0,S],\tag{8}$$

$$\left|F(t,\mathbf{u}(t))\right|\le L\,(1+|\mathbf{u}(t)|),\quad t\in[0,S].\tag{9}$$

Using (9), as a Cauchy-Schwarz inequality, we have

$$\mathbf{E}\left(\sup_{t\le S}\left|\int_0^t F(t_1,\mathbf{u}(t_1))dt_1\right|^2\right)\le\mathbf{E}\left(\sup_{t\le S}\int_0^t 1^2dt_1\int_0^t\left|F(t_1,\mathbf{u}(t_1))\right|^2dt_1\right)$$
$$=\mathbf{E}\left(S\int_0^t\left|F(t_1,\mathbf{u}(t_1))\right|^2dt_1\right)=S\int_0^t\mathbf{E}\left(\left|F(t_1,\mathbf{u}(t_1))\right|^2\right)dt_1$$
$$\le L^2S\int_0^t\mathbf{E}(1+|\mathbf{u}(t_1)|)^2dt_1=L^2S\int_0^t\left[1+2\mathbf{E}\,(|\mathbf{u}(t_1)|)+\mathbf{E}\,(\,|\mathbf{u}(t_1)|^2)\right]dt_1$$
$$\le L^2S\int_0^t\left[1+3\,\mathbf{E}\,(\,|\mathbf{u}(t_1)|^2)\right]dt_1\le L^2S\left[S+\int_0^t\mathbf{E}\left(\sup_{z\le t_1}|\mathbf{u}(z)|^2\right)dt_1\right].$$

Then use Doob's inequality, Itô integral and (9) in the second term above to arrive at

$$\mathbf{E}\left(\sup_{t\le S}\left|\int_0^t(-\sigma(t_1,\mathbf{u}(t_1)))dP(t_1)\right|^2\right)\le 4\,\mathbf{E}\left(\left|\int_0^t(-\sigma(t_1,\mathbf{u}(t_1)))dP(t_1)\right|^2\right)$$
$$\le 4\int_0^t\mathbf{E}\left(|\sigma(t_1,\mathbf{u}(t_1))|^2\right)dt_1\le 4L^2\int_0^t\mathbf{E}(1+|\mathbf{u}(t_1)|)^2dt_1$$
$$\le 4\,L^2\left[S+\int_0^t\mathbf{E}\left(\sup_{z\le t_1}|\mathbf{u}(z)|^2\right)dt_1\right].$$

Now we have

$$\mathbf{E}\left(\sup_{t \le S} |(\mathbf{u}(t)|^2 \right) \le \; 3\,\mathbf{E}(|\mathbf{u}_0|^2) \; + \; (S+4)\, L^2 \left[S + \int_0^t \mathbf{E}\left(\sup_{z \le t_1} |\mathbf{u}(z)|^2 \right) dt_1 \right],$$

and by means of Gronwall's lemma, we obtain

$$\mathbf{E}\left(\sup_{t \le S} |\mathbf{u}(t)|^2 \right) \le \left[3\,\mathbf{E}(|\mathbf{u}_0|^2) \; + \; (S^2 + 4)L^2 \right] \exp\,(S^2 + 4)L^2 t = C\,[\mathbf{E}(|\mathbf{u}_0|^2) + C_1],$$

where C and C_1 are constants. Therefore (5) has a strong solution and $\mathbf{E}\left(\sup_{t \le S} |\mathbf{u}(t)|^2 \right) \le +\infty$. Now, we prove uniqueness for the strong solution. Assume $\hat{\mathbf{u}}$ to be another solution vector (5) with the same σ, F, P and the same initial value vector $\mathbf{u}_0$. Use then (7) and (8) to obtain the same inequalities

$$\mathbf{E}\left(\sup_{t \le S} \left| \hat{\mathbf{u}}(t) - \mathbf{u}(t) \right|^2 \right) \le C(L,S)\;\; \mathbf{E}\left(\left| \mathbf{u}_0 - \mathbf{u}_0 \right|^2 \right) = 0,$$

where $C(L,S)$ is a constant. Hence $\mathbf{u}(t)$ is a unique strong solution vector to (5). ■

3. NURBS Elements

Here we present abbreviated piecewise Bernestein polynomial, Bézier elements and B-spline then we combine these concepts to obtain non uniform rational B-spline basis (NURBS). The $p+1$ Bernstein basis polynomial of degree p are defined for $x \in [0,1]$ as

$$\mathfrak{B}_{i,p}(x) = \begin{pmatrix} p \\ i \end{pmatrix} x^i\,(1-x)^{p-i},\; i = 0, \ldots\ldots, p. \tag{10}$$

These constitute a basis of the polynomials of degree p are pointwise non-negative. The incentive for performing finite element computation using this basis come from the fact that a piecewise Bernestein polynomial basis can be mapped onto a B-spline basis by invoking the Bézier extraction operator (see [15]).

This transformation enables the representation of a NURBS or a T-spline by using a set of Bézier elements. Therefore, we consider a degree p Bézier curve that it is defined by a linear combination of $p+1$ Bernstein polynomial basis functions. We define the set of basis functions as $B(x) = \{\mathfrak{B}_{i,p}(x)\}_{i=1}^{p+1}$, and the corresponding set of vector-valued control points as $P = \{p_i\}_{i=1}^{p+1}$, where each $p_i \in \mathbb{R}^d$, d being the number of spatial dimensions, and P is a matrix of dimension $(p+1)d$, viz $P = \left\{p_i^j\right\}_{i,j=1}^{p+1,\,d}$. Hence, the Bézier curve can then be written as:

$$R_p(x) = \sum_{i=1}^{p+1} p_i \mathfrak{B}_{i,p}(x) = P^T\,\mathfrak{B}(x),\;\; x \in [0,1].$$

A univariate B-spline basis is defined by a knot vector. The knot vector is a set of non-decreasing parametric coordinates written as $\Lambda = \left\{\zeta_1, \zeta_2, \ldots\ldots, \zeta_{n+p+1}\right\}$, where $\zeta_A \in \mathbb{R}$ is the Ath knot, p is the polynomial degree of the B-spline basis functions, and n is the number of basis functions. We define B-spline basis functions of degree p, recursively over the parametric domain by the knot vector. Piecewise constants are first defined as

$$N_{A,0}(\zeta) = \begin{cases} 1\ ,\ \zeta_A \leq \zeta < \zeta_A + 1, \\ 0, otherwise. \end{cases}$$

For $p > 0$, the basis functions are defined by the Cox-de Boor recursion formula

$$N_{A,n}(\zeta) = \frac{\zeta - \zeta_A}{\zeta_{A+n} - \zeta_A}\ N_{A,\,n-1}(\zeta) + \frac{\zeta_{A+n+1} - \zeta}{\zeta_{A+n+1} - \zeta_A\ +1}\ N_{A-1,\,n-1}(\zeta).$$

A B-spline curve of degree p in $\mathbb{R}^d$ is defined by a set of B-spline basis functions, $N(\zeta) = \{N_{A,n}(\zeta)\}_{A=1}^{l}$, and control points, $P = \{p_A\}_{A=1}^{l}$ as

$$T(t) = \sum_{A=1}^{l} p_A\, N_{A,n}(\zeta) = P^T N(\zeta).$$

By using a knot vector $\Lambda = \{\zeta_1, \zeta_2, \ldots\ldots, \zeta_{n+p+1}\}$ a set of rational basis functions $R = \{R_{A,n}\}_{A=1}^{l}$ and a set of control points $P = \{p_A\}_{A=1}^{l}$, we may refine an NURBS as

$$T(\zeta) = \sum_{A=1}^{l} p_A\, R_{A,n}(\zeta).$$

The NURBS basis functions (illustrated by the sketch of Fig.1) are defined as

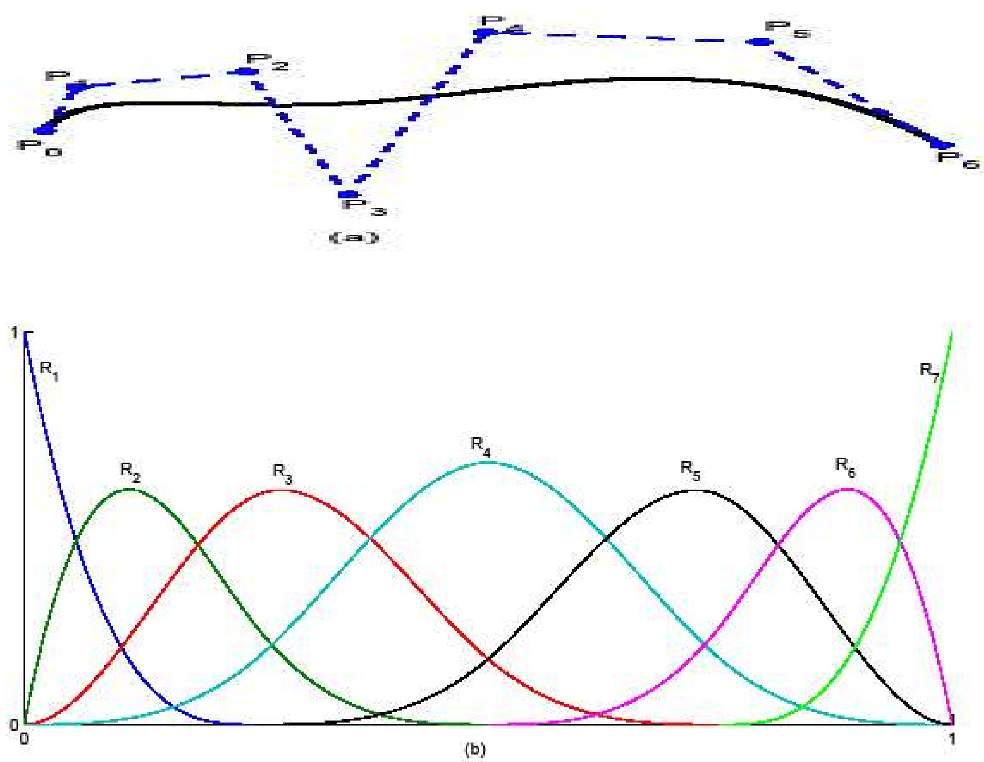

Figure1: A cubic NURBS curve; (a) the curve and its control net and (b) the basis functions of the curve. The knot vector for the curve is {0,0,0,0,1/4,2/4,3/4,1,1,1,1}.

$$R_{A,n}(\zeta) = \frac{\omega_A \; N_{A,n}(\zeta)}{W(\zeta)},$$

where

$$R_{A,n}(\zeta) = \sum_{B=1}^{l} \omega_B \; N_{B,n}(\zeta).$$

4. The New DG Method for a Stochastic Adsorption-Desorption Problem

In this section, we apply hybrid NURBS elements and discontinuous Galerkin method for mass-spring systems on $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$ that is a filtered probability space. We define $L_2(\Omega, H)$ to be the space of H −valued square integrable with norm:

$$\|\mathbf{u}\|^2_{n,\,L_2(\Omega,H)} = (\mathbf{u},\mathbf{u})_n = \mathbf{E}\left(\|\mathbf{u}\|^2_H\right) = \int_\Omega \int_{I_n} \|\mathbf{u}(t,\omega)\|^2 \, dt \, dP(\omega), n = 0,1,2,\ldots\ldots,N-1.$$

where H is separable Hilbert space and $u(.,.)$ is a H −valued square integrable function, so we define $\|\mathbf{u}\|^2_H$. For simplify, we consider the DG-method for (4) is based on using finite element over time. To define this method, let $0 = t_0 < t_1 <\ldots\ldots, t_N = S$ be a subdivision of the time interval $[0,S]$ into intervals $I_n = (t_n \, , \, t_{n+1})$, with time steps $k_n = t_{n+1} - t_n$, $n = 0,1,2,\ldots\ldots,N-1$. Further, for each n let $\mathbf{U}^n$ be a finite element subspace of $H^1(I_n) \times H^1(I_n)$, (we consider Sobolev space $H^1(I_n) = \{u \mid u,\, u_t \in L_2(I_n)\}$, see [1]). Then, we formulate DG-method on the interval I_n for (4), as follows. For $n = 0,1,2,\ldots\ldots,N-1$, find $\mathbf{u} \in \mathbf{U}^n$ such that

$$- < \mathbf{u} \, , \mathbf{v}^{\cdot} >_n \; + \; < A\mathbf{u}, \mathbf{v} >_n \; + [\mathbf{u}^n]^T \bullet \mathbf{v}^n_+ \; = \; < F, \mathbf{v} >_n \; , \forall \;\; \mathbf{v} \in \mathbf{U}^n \, , \qquad (11)$$

with

$$\mathbf{v}(t)_+ = \lim_{s \to 0^+} \mathbf{v}(t+s) \, , \quad \mathbf{v}(t)_- = \lim_{s \to 0^-} \mathbf{v}(t+s) \, .$$

We assume $\mathbf{u}^n = \mathbf{u}(t_n)$ and $h = \max_n k_n$ and we have test functions of the form $\mathbf{v}$. Further, we define the following notations for (5):

$$< \mathbf{u}, \mathbf{v} >_n = \int_{I_n} \mathbf{u}^T \bullet \mathbf{v} \, dt \, , \; n = 0,1,2,\ldots\ldots,N-1.$$

More concisely, after summing over n, taking all the slabs together, we define the function space $\mathbf{U} = \oplus_{n=0}^{N-1} \mathbf{U}^n$. Therefore, we may rewrite (5) as follow:

Find $\mathbf{u} \in \mathbf{U}$, such that

$$-B(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \;\; , \forall \;\; \mathbf{v} \in \mathbf{U}^n \, , \qquad (12)$$

where the bilinear form $B(.,.)$ and linear form $L(.)$ are respectively defined by by:

$$-B(\mathbf{u},\mathbf{v}) = \sum_{n=0}^{N-1}\left(- < \mathbf{u}, \mathbf{v}^{\cdot} >_n \; + \; < A\mathbf{u}, \mathbf{v} >_n \right) + \sum_{n=1}^{N-1} [\mathbf{u}^n]^T \bullet \mathbf{v}^n_+ \, ,$$

$$L(\mathbf{v}) = \sum_{n=0}^{N-1} < F, \mathbf{v} >_n \, .$$

We suppose that $\mathbf{u}^0_- = \mathbf{0}$. Also, we define the jump $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)^T$ in point $t_i$such that if $i = 1,2$, then we have:

$[\mathbf{u}_i] = \mathbf{u}_{i,+} - \mathbf{u}_{i,-}$, $[\mathbf{u}] = ([\mathbf{u}_1],[\mathbf{u}_2])^T$.

Now, we assume that $\mathbf{U}_h^n$ is the space of continuous piecewise polynomial functions of degree k as it be a subspace of $H^1(I_n) \times H^1(I_n)$, and sum over n, taking all the slabs together, to obtain

$\mathbf{U}_h = \oplus_{n=0}^{N-1} \mathbf{U}_h^n$.

Now, we assume that $\mathbf{u}_h \in \mathbf{U}_h$ is a solution of (5) , thus (6) can be formulated as follows:

$$-B(\mathbf{u}_h, \mathbf{v}) = L(\mathbf{v}) \ , \tag{13}$$

for $\mathbf{v} \in \mathbf{U}_h$. If we define $\varepsilon = \mathbf{u} - \mathbf{u}_h$ such that $\mathbf{u} \in \mathbf{U}$ is an exact solution of (4), then we have the following error equation

$$-B(\varepsilon, \mathbf{v}) = 0 \ . \tag{14}$$

5. An a Posteriori Error Estimate for the DG-Method

In this section, we obtain a posteriori error for (4) when $(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{F}_t)$ is a filtered probability space . To derive a posteriori error estimates for the scheme (4), we define a dual problem for it as follows. Find $\phi(t) = (\phi_1(t), \phi_2(t))^T$ such that

$$\left.\begin{array}{l} -\ \phi(t) + A^T(t)\ \phi(t) = 0, \ \ t \in [0, S], \\ \phi(t_N) = \varepsilon^N, \end{array}\right\} \tag{15}$$

where, by duality, $A^T$is considered instead of A. If $\mathbf{E}\left(|\phi_1(t)|^2\right) < +\infty$ and $\mathbf{E}\left(|\phi_2(t)|^2\right) < +\infty$, then (4) has a unique solution satisfying $\mathbf{E}\left(\underset{t \le S}{sup}\ \|\phi(t)\|^2_{n, L_2(\Omega, H)}\right) < +\infty$.

Theorem 5.1. *For the error* $\varepsilon = \mathbf{u} - \mathbf{u}_h$, *where u is the solution of the continuous problem (4) and* u_h *is the solution of (7), the stability estimate*

$$-\mathbf{E}(\|\mathbf{u}(t_N) - \mathbf{u}_h(t_N)\|_H) \ \ \le C\, h\ \gamma_{t_N}\ \mathbf{E}(R(\mathbf{u})), \tag{16}$$

always holds, when

$$R(\mathbf{u}) = k_n \|R_1\|_H \ + \|R_2\|_H \ , \tag{17}$$

and γ_{t_N} *is the associated stability factor:*

$$-\gamma_{t_N} = \frac{\int_0^S \|\phi^{\cdot}\|_{\infty,[0,S]}\, dt}{\|\varepsilon^N\|_H}, \tag{18}$$

with $\phi = (\phi_1, \phi_2)^T$, $\phi^{\cdot} = \frac{d}{dt}\phi$, *and* $\|\phi\|_{\infty,[0,S]} = \underset{[0,S]}{max}\ (|\phi_1|, |\phi_2|)$, *when* $\|\phi\|_\infty = \frac{1}{2}(|\phi_1| + |\phi_2|)$.

Proof. Noting that $\varepsilon = \mathbf{u}\ - \mathbf{u}_h$, $\mathbf{u} \in \mathbf{U}$,$-\mathbf{u}_h \in \mathbf{U}_h$ and $\|\varepsilon^N\|_H^2 = \varepsilon^{N^T} \cdot \varepsilon^N$, then consider $\varepsilon^{\cdot}$ $+ A\varepsilon = u^{\cdot} - u_h^{\cdot} + Au - Au_h = f - (u_h - Au_h)$ in (4) to obtain

$$\mathbf{E}\left(\|\varepsilon^N\|_H^2\right) = <0, \varepsilon>_{[0,S]}$$
$$= \sum_{n=0}^{N-1} <-\phi^{\cdot} + A^T\phi,\ \varepsilon>_n$$
$$= \sum_{n=0}^{N-1}\left(<\ \varepsilon^{\cdot} + A^T\varepsilon,\ \phi>_n - [\varepsilon\phi]_{t_n}^{t_{n+1}}\right)$$
$$= \sum_{n=0}^{N-1}\left(<f-(u_h + A^T u_h),\ \phi>_n - [\varepsilon\phi]_{t_n}^{t_{n+1}}\right). \tag{19}$$

Consider, moreover, $q(t_{n+1}^-) = q_{n+1}^-$, $q(t_n^+) = q_n^+$ to represent the second sum in (19) by

$$J = \sum_{n=0}^{N-1}[\varepsilon^T\phi]\ |_{t_n}^{t_{n+1}} = \sum_{n=0}^{N-1}\varepsilon^T(t_{n+1}^-)\,\phi(t_{n+1}^-) - \varepsilon^T(t_n^+)\,\phi(t_n^+)$$
$$= (\varepsilon^T\bullet\phi)_-^1 - (\varepsilon^T\bullet\phi)_+^0 + (\varepsilon^T\bullet\phi)_-^2 - (\varepsilon^T\bullet\phi)_+^1$$
$$+\ldots\ldots + (\varepsilon^T\bullet\phi)_-^{N-1} - (\varepsilon^T\bullet\phi)_+^{N-2} + (\varepsilon^T\bullet\phi)_-^N - (\varepsilon^T\bullet\phi)_+^{N-1}$$

We continue by writing $\phi_-^n = \phi_-^n - \phi_+^n + \phi_+^n$, $n = 0,1,2,\ldots\ldots,N-1,$ to conclude that

$$-J = -(\varepsilon^T\bullet\phi)_-^N + (\varepsilon^T\bullet\phi)_+^0 + \sum_{n=1}^{N-1}[\varepsilon^n]^T\bullet\phi_+^n \div \sum_{n=1}^{N-1}[\phi^n]^T\bullet\varepsilon_-^n\,.$$

Also, we define the jump $\phi = (\phi_1,\phi_2)^T$ in point t_i such that if $i = 1,2$, then we have:
$[\phi_i] = \phi_{i,+} - \phi_{i,-}$, $[\phi] = ([\phi_1] + [\phi_2])^T$.

According to (4), $\phi(t_N) = \varepsilon^N$ and since $\varepsilon(t_0)_- = 0,$ ϕ and u are smooth $[\phi^n] = [u^n] = 0,$ $[\varepsilon^n] = -[u_h^n].$ Hence, we get

$$-J = \sum_{n=0}^{N-1}[\mathbf{u}_h^n]\bullet\phi_+^n + \|\varepsilon^N\|_2^2. \tag{20}$$

Then from (19), by use (20), we arrive at

$$\mathbf{E}\left(\|\varepsilon^N\|_H^2\right) = \sum_{n=0}^{N-1} <f-(u_h + A^T u_h),\ \phi>_n - \sum_{n=0}^{N-1}\mathbf{E}\left([\varepsilon u_h^n]^T\phi_+^n\right).$$

Further use of the interpolation $\pi_n\,\phi = \left(\pi_n\,\phi_1, \pi_n\,\phi_2\right)^T$ of $\phi = (\phi_1,\phi_2)^T$ yields

$$\mathbf{E}\left(\|\varepsilon^N\|_H^2\right) = \sum_{n=0}^{N-1} <f-(u_h + A^T u_h),\ (\phi-\pi_n\phi)>_n - \sum_{n=0}^{N-1}\mathbf{E}\left([\varepsilon u_h^n]^T(\phi-\pi_n\phi)_+^n\right). \tag{21}$$

The idea is now to estimate $\Phi - \phi$ by using a strong stability estimates for solution ϕ of the dual problem.

$$\int_{I_n}\left|\phi - \pi_n\,\phi\right| dt \ \le\ k_n\int_{I_n}\left|\,\phi^{\cdot}\right| dt,$$

$$\|\phi - \pi_n\,\phi\|_{\infty, I_n} \le \sum_{n=0}^{N-1} k_n\ \|\phi^{\cdot}\|_{\infty, I_n}\,.$$

Let $\phi = \frac{1}{k_n} \int_{t_{n+1}} \phi(s)\, ds$ be the mean value of ϕ over I_n, then by using the mean value theorem for integration we can write

$$\int_{t_n}^{t_{n+1}} \left|\phi - \pi_n \phi\right| dt = \int_{t_n}^{t_{n+1}} |\phi(t) - \phi(\theta)|\, dt = \int_{t_n}^{t_{n+1}} \left|\int_\theta^t \phi(s) ds\right| dt$$

$$\leq \int_{t_n}^{t_{n+1}} \left|\int_{t_n}^{t_{n+1}} \phi(s) ds\right| dt = k_n \int_{I_n} \left|\phi^\cdot\right| dt\,, \theta \in I_n. \tag{22}$$

Now, we define residuals of computed solution u_h by

$$R_1 = \mathbf{u}_h^\cdot - A\mathbf{u}_h \,, \quad \text{in } \mathbf{U}_h,$$

$$R_2 = \mathbf{u}_{h,+}^n - A\mathbf{u}_{h,-}^n \,, \quad \text{in } \mathbf{U}_h. \tag{23}$$

Then using (22)} and (23) leads to

$$\mathbf{E}\left(\|\varepsilon^N\|_H^2\right) = \sum_{n=0}^{N-1} < f - (u_h + A^T u_h),\ (\phi - \pi_n\phi) >_n - \mathbf{E}\left([\varepsilon u_h^n]^T (\phi - \pi_n\phi)_+^n\right)$$

$$\leq c \sum_{n=0}^{N-1}\left(\|R_1\|_H + \frac{1}{k_n}\|R_2\|_H\right) k_n \int_{I_n} \|\phi^\cdot\|_{\infty, I_n}\, dt$$

$$\leq \max_n \left(c\, k_n \|R_1\|_H + \|R_2\|_H \right) \int_0^S \|\phi^\cdot\|_{\infty, [0,S]}\, dt.$$

Therefore we obtain

$$\|\varepsilon^N\|_H^2 \leq h\, \gamma_{t_N} \|\varepsilon^N\|_H \ \|R(u_h)\|_H\,,$$

$$\|\varepsilon^N\|_H \leq h\, \gamma_{t_N} \ \|R(u_h)\|_H\,.$$

■

6. Convergence Analysis

$\varepsilon = u - u_h$

In this section we present a theorem for estimate older of convergence for this end we introduce two notations to estimate $\varepsilon = u - u_h$. We set $\eta = u - \mathfrak{R}_h\, u$, $\xi = u_h - \mathfrak{R}_h\, u$, where $\mathfrak{R}$ is an L_2 projection. Thus we have $\varepsilon = (u - \mathfrak{R}_h\ u) - (\mathfrak{R}_h u - u_h) = (\eta - \xi)$, then recall the Galerkin orthogonality relation $B(\varepsilon, v)$.

Lemma 6.1 (Interpolation inequality). *For any $v \in W^{k+1,\infty}(\Omega)$, $(\Omega + [0,S])$ there exist a $C > 0$ (constant) independent of h, such that*

$$\|(v - \mathfrak{R}v)\| \leq c\, h^{k+1} |v|_{H^{k+1}(\Omega)}\,. \tag{24}$$

Corollary 6.1. *A consequence of the assumption in Lemma 6.1 is* $\|\eta(t)\| \leq c\, h^{k+1} |u(t)|_{H^{k+1}}$.

Theorem 6.1. *Let u be the exact solution of (4), if $u_h \in U_h$ is the numerical solution of the DG method, then there is a constant c such that*

$$\|u - u_h\| \leq c\, h^{k+1} |u|_{H^{k+1}(\Omega)}\,. \tag{25}$$

Proof. It is clear that $u(t)$ satisfies the weak form. Hence, we can write the error equations as follows.

$$B(\varepsilon, v) = 0,$$

$$\sum_{n=0}^{N-1} - < \varepsilon, v^{\cdot} >_n + < A\varepsilon, v>_n + \sum_{n=1}^{N-1}[\varepsilon^n]^T v_+^n = 0. \qquad (26)$$

By setting $v = \xi$ and $\varepsilon = \eta - \xi$ we have

$$\sum_{n=0}^{N-1} \left(- < \eta - \xi,\ \xi^{\cdot} >_n + < A(\eta - \xi),\ \xi >_n\right) + \sum_{n=1}^{N-1}[(\eta - \xi)^n]^T \xi_+^n = 0.$$

$$\sum_{n=0}^{N-1} \left(- < \eta,\ \xi^{\cdot} >_n - < \xi,\ \xi^{\cdot} >_n + < A\eta,\ \xi >_n - < \xi,\ \xi >_n\right)$$

$$+ \sum_{n=1}^{N-1}[\eta^n]^T \xi_+^n - \sum_{n=1}^{N-1}[\xi^n]^T \xi_+^n = 0.$$

$$\Rightarrow D_1 + D_2 + D_3 + D_4 + D_5 + D_6 = 0. \qquad (27)$$

We continue by estimating each of the D terms. For D_2, D_4, D_6 it is easy to show that $|D_2| \le (1/2)\|\xi\|^2$, $|D_4| \le \|\xi\|^2$, $|D_6| \le \|\xi\|^2$. Then, by applying Young's inequality and Corollary 6.1, we have:

$$|D_1| = \left|\sum_n < \eta, \xi^{\cdot} >_n\right| \le \sum_n c\, h_n - 1\ \|\xi\|_{L^2(I_n)}\ \|\eta\|_{L^2(I_n)} \le c\, h^{-1}\|\xi\|\ .\ c\, h^{k+1}\ |u(.)|_{H^{k+1}}$$

$$\le c\, h^{-2}\|\xi\|^2 + c\, h^{2k+2}\ |u(.)|_{H^{k+1}}^2 \quad \le c\ \|\xi\|^2 + c\, h^{2k+2}\ |u(.)|_{H^{k+1}}^2\ ,$$

$$|D_3| = \left|\sum_n < A\eta, \xi >_n\right| \le \sum_n \|A\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)}\ \|\eta\|_{L^2(I_n)} \le \|\xi\|\ .\ c\, h^{k+1}\ |u(.)|_{H^{k+1}}$$

$$\le c\ \|\xi\|^2 + c\, h^{2k+2}\ |u(.)|_{H^{k+1}}^2\ ,$$

$$|D_5| = \left|\sum_n \left|[\eta^n]^T \xi^n\right|\right| \le \sum_n \|\xi\|_{L^2(I_n)}\ \|\eta\|_{L^2(I_n)} \le \|\xi\|\ .\ c\, h^{k+1}\ |u(.)|_{H^{k+1}}$$

$$\le c\ \|\xi\|^2 + c\, h^{2k+2}\ |u(.)|_{H^{k+1}}^2\ .$$

Then we consider throughout $\|\xi\| \le c\ h^{k+1}\ |u(.)|_{H^{k+1}}$. To complete this proof, we use inverse inequalities and

$$\|\xi\| \le ch^{k+1}|u(.)|_{H^{k+1}} \le ch^k\ \|u\|_{H^{k+1}}.$$

■

7. Numerical Tests

In this section we demonstrate that the Discontinuous Galerkin method for the mass spring system of (3) with Poisson distribution for $f(t, x(t))$. Therefore, we carry out the following two algorithms. An AMILO computer with 15 Gigabytes RAM memory and 2.2 GHz CPU has been used for these experiments. In the first algorithm, we develop some ODE solvers in MATLAB. Clearly, there is no exact solution available from a numerical viewpoint. Also, we use the sample of Monte Carlo method for (25) and (26) as a criterion. In the second algorithm, we use the Euler Scheme (ES) for (3).

Algorithm 7.1. The Stochastic Discontinuous Galerkin (SD) scheme for (3)
Step **0**-Start
Step **1**- Input c , $m_0, m_a, \lambda, T, S, \beta, \alpha, N$ and δ.
Step **2**- Discretization of $[0,S]$, $0 = t_0 < t_1 < \ldots, t_N = S$, $h_n = t_{n+1} - t_n$ and $h = max\, h_n \leq 1$.
Step **3**- $P(T) = Poispdf^*(T,\lambda) = \frac{\lambda^T e^{-\lambda}}{T!} I_{(0,1,2,3,\ldots)}(T), T \in \mathbb{Z}^+$.
Step **4**- Solve equation (12) by DG for the following initial conditions:
$x(0), x'(0)$,
or
$0 \leq \mathbf{E}\left(|x(0)|^2\right) \leq \beta,\ 0 \leq \mathbf{E}\left(|x'(0)|^2\right) \leq \gamma$.
Step **5**- Plot $\left|x^{SD}_{h,N} - x^{SD}_{h,N-1}\right|$, where $x^{SD}_{h,N}$ is an approximation solution by *SD*.
Step **6**- Estimator $\mathbf{E}(R(u)) = \int_0^S R(u(\omega))\ dP(\omega) \simeq \frac{1}{N} \sum_{i=0}^{N} R(u(\omega_i))$ by Monte Carlo (MC) simulation.
Step **7**- End.

To compare with the SD method, we give the following standard algorithm.

Algorithm 7.2. Euler scheme (ES) for (3)
Step **0**-Start
Step **1**- Input c , $m_0, m_a, \lambda, T, S, \beta, \alpha, N$ and δ.
Step **2**- Discretization of $[0,S]$, $0 = t_0 < t_1 < \ldots, t_N = S$, $h_n = t_{n+1} - t_n$ and $h = max\, h_n \leq 1$.
Step **3**- Euler's method: Consider the following iteration formula, if $x_1^N(0) = x_1(0)$ and $x_2^N(0) = x_2(0)$,

$$\left.\begin{aligned} & x_1^N(t_{i+1}) = x_1^N(t_i) + x_2^N(t_i)\, \triangle_i\, , \\ & x_2^N(t_{i+1}) = x_2^N(t_i) - \frac{c}{m_0 + m_a \times N(\varphi(t))}\, x_1^N(t_i)\, \triangle_i = f(t_i, x_1^N(t_i), x_2^N(t_i)), \\ & \triangle_i = t_{i+1} - t_i\ , \qquad i = 0,1,2,3,\ldots, N-1. \end{aligned}\right\} . \tag{28}$$

Step **4**- Plot $\left|x^{ES}_{h,N} - x^{ES}_{h,N-1}\right|$, where $x^{ES}_{h,N}$ is an approximation solution by *ES*.
Step **5**- End.

We use the Poisson distribution in Fig 2 where we compare $\left|x^{SD}_{h,N} - x^{SD}_{h,N-1}\right|$ with $\left|x^{ES}_{h,N} - x^{ES}_{h,N-1}\right|$. The computed results show that SD is more efficient. Finally, Fig 3 exhibits estimates of R(u) by the MC (see [9]).

*Poisson probability density function

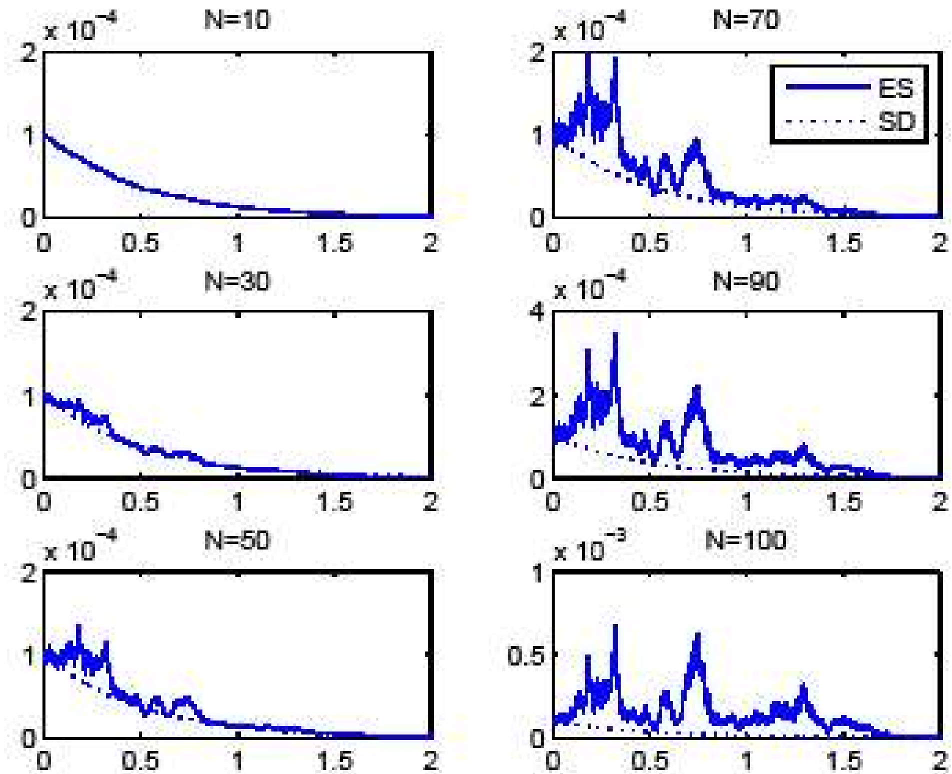

Figure 2: Comparison of $\left|x_{h,N}^{SD} - x_{h,N-1}^{SD}\right|$ with $\left|x_{h,N}^{ES} - x_{h,N-1}^{ES}\right|$ for various N and the parameters $\alpha = \beta = c = m_0 = m_a = S = 2$, $\delta = 0$.

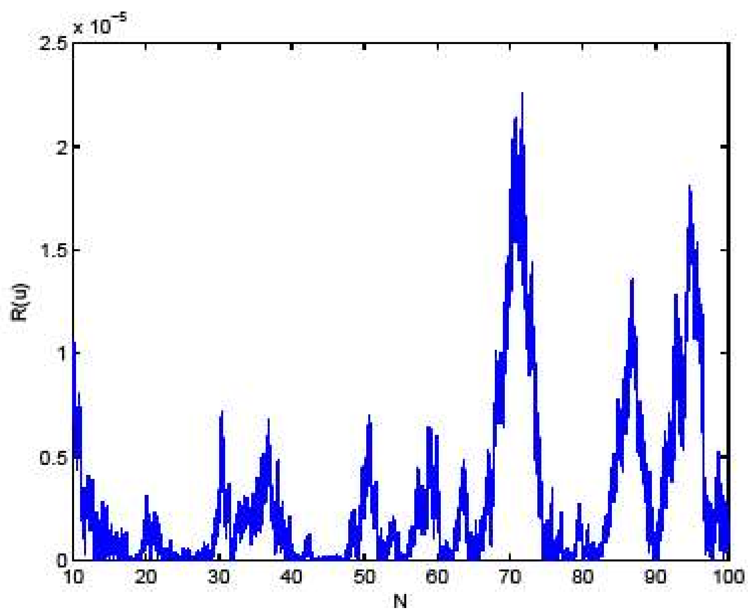

Figure 3: MC simulation for $R(u)$.

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