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Homogenization of a Degenerate PDE with a Nonlinear Neumann Boundary Condition

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Abstract. *We establish homogenization results of a degenerate semilinear PDE with a nonlinear Neumann boundary condition. Our approach is entirely probabilistic, and extends the result of Pardoux and Ouknine published in 2002.*

Key words: Homogenization, Viscosity Solution, Backward Stochastic Differential Equations, Neumann Boundary Condition.

AMS Subject Classifications: 60H15, 35K20, 35B27

1. Introduction

We study the following semi-linear partial differential equation (PDE) in a domain $D = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$, with a non-linear Neumann boundary condition on ∂D. For each $\varepsilon > 0$, we consider

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon\, u^\varepsilon(t,x) + f\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) + \frac{1}{\varepsilon} e\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right),\ x \in D,\ 0 < t, \\ \Gamma_\varepsilon\, u^\varepsilon(t,x) + h\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) = 0,\ x \in D,\ 0 < t, \\ u^\varepsilon(0,x) = g(x),\ x \in D, \end{cases} \tag{1}$$

where

• $e : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a measurable mapping, which is periodic, of period one in each direction in the first argument, continuous in the second argument, uniformly with respect to the first, and satisfies :

$$\int_{\mathbb{T}^d} e(x,y)m(dx) = 0, \qquad \forall y \in \mathbb{R}, \tag{2}$$

when m is the unique invariant measure on the torus T^d.

Suppose e to be twice continuously differentiable in y, uniformly with respect to x, and there exists a constant K such that :

$$|e(x,y)| + \left|\frac{\partial}{\partial y}e(x,y)\right| + \left|\frac{\partial^2}{\partial y^2}e(x,y)\right| \leq K, \quad \forall x \in \mathbb{R}^d,\ y \in \mathbb{R}; \tag{3}$$

• $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are sufficiently smooth functions. Equivalently the coefficients can be seen as periodic functions with respect to the first variable with period one in each direction on $\mathbb{R}^d$ and are such that for some $c > 0$, $p > 0$, $\mu \in \mathbb{R}$, $\beta < 0$, and all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$:

$$|g(x)| \leq c(1 + |x|^p), \tag{4}$$

$$|f(x,y)| \leq c\left(1 + |y|^2\right), \tag{5}$$

$$(y - y')[f(x,y) - f(x,y')] \leq \mu|y - y'|^2, \tag{6}$$

$$(y - y')[h(x,y) - h(x,y')] \leq \beta|y - y'|^2, \tag{7}$$

$$|h(x,y)| \leq c\left(1 + |y|^2\right). \tag{8}$$

2. Assumptions and Definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a d-dimensional Brownian motion $(B^1, \ldots, B^d)$ is defined. Let E the corresponding expectation operator.
The differential operator L_ε inside D is given by :

$$L_\varepsilon = \frac{1}{2}\sum_{i,j}^{d} a_{i,j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\varepsilon}\sum_{i=1}^{d} b_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}. \tag{9}$$

This operator is the generator of the reflected $(L_\varepsilon, \Gamma_\varepsilon)$-diffusion (see Tanaka [7]) :

$$\begin{cases} dX_t^\varepsilon = \sigma\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dB_t + \frac{1}{\varepsilon}b\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dt + c\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dt + \gamma\left(\frac{X_t^\varepsilon}{\varepsilon}\right)d\varphi_t^\varepsilon,\ 0 < t, \\ X^{1,\varepsilon} \geq 0, \varphi^\varepsilon \text{ is continuous and increasing, } \int_0^t X_s^{1,\varepsilon}\, d\varphi_s^\varepsilon = 0, \qquad 0 < t, \\ X_0^\varepsilon = x, \end{cases} \tag{10}$$

where $X^{1,\varepsilon}$ denotes the first component of the process X^ε. We recall that $D = \mathbb{R}_+^* \times \mathbb{R}^{d-1}$, so that X^ε lives is $\overline{D}$, that is $X^{1,\varepsilon}$ remains non-negative and φ^ε increases when and only when $X^{1,\varepsilon}$ is zero, just to keep it non-negative.
The associated differential operator on ∂D is defined as

$$\Gamma_\varepsilon := \sum_{i=1}^{d} \gamma_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}. \tag{11}$$

The function $\gamma : \partial D (\cong \mathbb{R}^{d-1}) \to \mathbb{R}^d$ is smooth and periodic of period one in each direction and satisfies : $\gamma^1(x) = 1$.
We suppose that $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \to \mathbb{R}^d$ are smooth and periodic of period one in each direction. The novelty here is that : the matrix $a = \sigma\sigma^*$ is degenerate and satisfies Hörmander's condition.

Definition 2.1 (Lie bracket). The Lie bracket between the vectors fields A_j and A_k is defined by :

$$[A_j, A_k] := A_j^{\nabla} A_k - A_k^{\nabla} A_j, \tag{12}$$

where $A_j^{\nabla} A_k := A_j^l \partial_l A_k^i \dfrac{\partial}{\partial x_i}$.

Definition 2.2 (Strong Hormander condition (SHC)). Let $H(n,x)$ be the set of the Lie brackets of $(\sigma_j(x))_{1\leq j\leq d}$ of order lower than n at the point $x \in \mathbb{R}^d$. We say that matrix σ satisfies the (SHC) if for all $x \in \mathbb{R}^d$, there exist $n_x \in \mathbb{N}$ such that $H(n_x,x)$ generates $\mathbb{R}^d$.

Let us define $\tilde{X}_t^{\varepsilon} := \frac{1}{\varepsilon} X^{\varepsilon}_{\varepsilon^2 t}$ and $\tilde{\varphi}_t^{\varepsilon} := \frac{1}{\varepsilon}\varphi^{\varepsilon}_{\varepsilon^2 t}$, then we get with a new standard d-dimensional Brownian motion $\left\{B_t^{\varepsilon} : t \geq 0\right\}$, which in fact depends on ε:

$$\begin{cases} d\tilde{X}_t^{\varepsilon} = \sigma(\tilde{X}_t^{\varepsilon})dB_t^{\varepsilon} + b(\tilde{X}_t^{\varepsilon})dt + \varepsilon c(\tilde{X}_t^{\varepsilon})dt + \gamma(\tilde{X}_t^{\varepsilon})d\tilde{\varphi}_t^{\varepsilon}, \qquad 0 < t, \\ \tilde{X}_t^{1,\varepsilon} \geq 0, \tilde{\varphi} \text{ is continuous and increasing, } \int_0^t \tilde{X}_s^{1,\varepsilon}\, d\tilde{\varphi}_s^{\varepsilon} = 0,\ 0 < t, \\ \tilde{X}_0^{\varepsilon} = \frac{x}{\varepsilon}. \end{cases} \tag{13}$$

From now on, we consider the T^d-values process $\tilde{X}$ whose operator is given by :

$$\tilde{L}_{\varepsilon} := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + \varepsilon \sum_{i=1}^{d} c_i(x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{T}^d. \tag{14}$$

This operator converges to L_0, given by

$$L_0 := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{T}^d. \tag{15}$$

Hypothesis (**H**): we suppose that σ satisfies the SHC.

3. Weak Convergence (SDE and BSDE)

With (H) and the boundary condition $\Gamma_0 u = 0$, below in (23), according to Tanaka [7] (see for example Pardoux and Diedhiou [5] in the case that $\partial D = \emptyset$), one can observe that the T^d-values $\tilde{X}^{\varepsilon}$, is an homogeneous Feller process with values in a compact set, and this process has a unique invariant measure whose density is strictly positive. Thereafter, we set m as the invariant measure associated of L_0 on T^d and we set m_0 as the invariant measure associated of the differential operator Γ_0 on T^{d-1} .

Throughout, we suppose that

$$\int_{\mathbb{T}^d} b(x)\, m(dx) = 0. \tag{16}$$

Consider $\hat{b}$ to be the solution of Poisson equation : $L_0\hat{b} = -b$, and let us introduce the process $\hat{X}_t^{\varepsilon}$ defined as:

$$\begin{aligned} \hat{X}_t^{\varepsilon} &= X_t^{\varepsilon} + \varepsilon\left[\hat{b}\left(\frac{X_t}{\varepsilon}\right) - \hat{b}\left(\frac{x}{\varepsilon}\right)\right] \\ &= x + \int_0^t \left(I + \nabla\hat{b}\right)\sigma\left(\frac{X_s^{\varepsilon}}{\varepsilon}\right) dB_s + \int_0^t \left(I + \nabla\hat{b}\right)c\left(\frac{X_s^{\varepsilon}}{\varepsilon}\right)ds \\ &\quad + \int_0^t \left(I + \nabla\hat{b}\right)\gamma_{\varepsilon}\left(\frac{X_s^{\varepsilon}}{\varepsilon}\right) d\varphi_s^{\varepsilon}. \end{aligned} \tag{17}$$

We may write (17) in coordinate form :

$$\begin{cases} \hat{X}_t^{1,\varepsilon} = x_1 + \int_0^t \left(I + \nabla \hat{b}\right) \sigma_1 \left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s^1 + \int_0^t \left(I + \nabla \hat{b}_1\right) c_1 \left(\frac{X_s^\varepsilon}{\varepsilon} \right) ds \\ \qquad + \int_0^t \left(I + \nabla \hat{b}_1\right) \left(\frac{X_s^\varepsilon}{\varepsilon} \right) d\varphi_s^\varepsilon, \\ \text{and} \\ \hat{X}_t^{j,\varepsilon} = x_j + \int_0^t \left(I + \nabla \hat{b}_j\right) \sigma_j \left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s^j + \int_0^t \left(I + \nabla \hat{b}_j\right) c_j \left(\frac{X_s^\varepsilon}{\varepsilon} \right) ds, \\ \forall\, j = 2, 3, \ldots, d. \end{cases} \tag{18}$$

By (H) we know that the pair $(L_0, \gamma \cdot \nabla)$ is solvable on $\overline{D}$ and L_0 is hypo-elliptic. Then there exists a bounded and smooth solution η of the PDE, with Neumann-type boundary condition,

$$\begin{cases} L_0 \eta = 0 & \text{in } \boldsymbol{D} \\ \gamma \cdot \nabla \eta = \left(1 + \nabla \hat{b}_1\right) - \int_{\mathbb{T}^{d-1}} \left(1 + \nabla \hat{b}_1\right)(x) m_0(dx) & \text{on } \partial \boldsymbol{D}. \end{cases} \tag{19}$$

Taking such a solution η, we have by Itô ,

$$\begin{aligned} \varepsilon \left[\eta \left(\frac{X_t^\varepsilon}{\varepsilon} \right) - \eta \left(\frac{x}{\varepsilon} \right) \right] &= \int_0^t \nabla \eta \sigma \left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s + \int_0^t \nabla \eta c \left(\frac{X_s^\varepsilon}{\varepsilon} \right) ds \\ &\quad + \int_0^t \left(1 + \nabla \hat{b}_1\right) \left(\frac{X_s^\varepsilon}{\varepsilon} \right) d\varphi_s^\varepsilon - \varphi_t^\varepsilon \int_{\mathbb{T}^{d-1}} \left(1 + \nabla \hat{b}_1\right)(x) m_0(dx). \end{aligned} \tag{20}$$

Putting (20) into the first component of (18) we have

$$\begin{aligned} \hat{X}_t^{1,\varepsilon} &= x_1 + \int_0^t \left(1 + \nabla \hat{b}_1\right) \sigma_1 \left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s^1 + \int_0^t \left(1 + \nabla \hat{b}_1\right) c_1 \left(\frac{X_s^\varepsilon}{\varepsilon} \right) ds \\ &\quad + \varphi_t^\varepsilon \int_{\mathbb{T}^{d-1}} \left(1 + \nabla \hat{b}_1\right)(x) m_0(dx) \\ &\quad \underbrace{- \int_0^t \nabla \eta \sigma \left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s - \int_0^t \nabla \eta c \left(\frac{X_s^\varepsilon}{\varepsilon} \right) ds + \varepsilon \left[\eta \left(\frac{X_t}{\varepsilon} \right) - \eta \left(\frac{x}{\varepsilon} \right) \right]}_{A_\varepsilon(t)} \end{aligned} \tag{21}$$

From (20), and in view of a Skorokhod's problem (see Pilipenko [6] or Tanaka [7]) it is straight forward to show that

$$\lim_{\varepsilon \to 0} \mathbb{E} \left\{ \max_{0 \le t \le T} |A_\varepsilon(t)| \right\} = 0. \tag{22}$$

Before proceeding, we introduce the definition :

$$\gamma_0 = \int_{\mathbb{T}^{d-1}} \left(I + \nabla \hat{b}\right)(x) \gamma(x) m_0(dx) \quad \text{and} \quad \Gamma_0 = \gamma_0 \cdot \nabla. \tag{23}$$

As in [3], we have the theorem that follows.

Theorem 3.1. *Under (H) and the condition (16), the* $(L_\varepsilon, \Gamma_\varepsilon)$*-reflected diffusion process* X^ε *converges in law to the* $\left(L, \Gamma_0\right)$*-reflected diffusion process* X *as* $\varepsilon \downarrow 0$. *Moreover, on the space* $C([0,T], \mathbb{R}^{2d+1})$ *equipped with the sup-norm topology,*

$(X^\varepsilon, M_t^{X^\varepsilon}, \varphi^\varepsilon) \to (X, M^X, \varphi)$

where

- $M_t^{X^\varepsilon} = \int_0^t \left(I + \nabla \hat{b}\right) \sigma \left(\frac{X_s^\varepsilon}{\varepsilon} \right) dB_s,$

• *M^X is the martingale part of X,*
• *φ (resp. φ^ε) is the local time of X^1 (resp. $X^{1,\varepsilon}$).*

Furthermore, let $\overline{X}$ denote the unique diffusion process with values in the d-dimensional torus $\mathbb{T}^d$, whose generator is the operator L_0. We now consider a type of BSDE which has been introduced in Pardoux and Zhang [2]. For each fixed $(t,x) \in [0,T] \times \overline{\boldsymbol{D}}$, let $\{(Y_s^\varepsilon, U_s^\varepsilon); 0 \le s \le T\}$ be the solution of the BSDE

$$\begin{aligned} Y_s^\varepsilon &= g(X_t^\varepsilon) + \int_s^t f\Big(\tfrac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\Big)dr + \frac{1}{\varepsilon}\int_s^t e\Big(\tfrac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\Big)dr \\ &\quad + \int_s^t h\Big(\tfrac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\Big)d\varphi_r^\varepsilon - \int_s^t U_r^\varepsilon dM_r^{X^\varepsilon}. \end{aligned} \tag{24}$$

For each fixed $y \in \mathbb{R}$, let the set $\hat{e}$ be the solution of the Poisson equation :

$$L_0\hat{e}(x,y) + e(x,y) = 0, \; x \in \mathbb{T}^d, \; y \in \mathbb{R}. \tag{25}$$

More precisely by (2), $\hat{e}$ is centered with respect to the invariant measure m and is given by the formula

$$\hat{e}(x,y) = \int_0^\infty \mathbb{E}^x e(\overline{X}_t, y)\, dt. \tag{26}$$

Note that, see [4], $\hat{e} \in C^{0,2}(\mathbb{T}^d, \mathbb{R})$ and $\hat{e}(.,y)$, $\frac{\partial}{\partial y}\hat{e}(.,y)$, $\frac{\partial^2}{\partial y^2}\hat{e}(.,y) \in W^{2,p}(\mathbb{T}^d)$, for any $p \ge 1$, there exists K' such that, for all $y \in \mathbb{R}$,

$$\|\hat{e}(.,y)\|_{W^{2,p}(\mathbb{T}^d)} + \left\|\frac{\partial}{\partial y}\hat{e}(.,y)\right\|_{W^{2,p}(\mathbb{T}^d)} + \left\|\frac{\partial^2}{\partial y^2}\hat{e}(.,y)\right\|_{W^{2,p}(\mathbb{T}^d)} \le K'. \tag{27}$$

We then introduce the notations,

$$M_t^\varepsilon = \int_0^t U_s^\varepsilon dM_s^{X^\varepsilon} \quad \text{and} \quad M_t = \int_0^t U_s dM_s^X, \quad 0 \le t \le T,$$

to consider the quintuple $\{X, M^X, \varphi, Y, M\}$ (*resp.* $\{X^\varepsilon, M^{X^\varepsilon}, \varphi^\varepsilon, Y^\varepsilon, M^\varepsilon\}$) as a random element of the space $C([0,t], \mathbb{R}^{2d+1}) \times D([0,t], \mathbb{R}^2)$, where we equip the first factor with the sup-norm topology, and the second factor with the S-topology of Jakubowski (see [1]).

Consider, moreover, the SDE and BSDE, satisfied respectively by X and Y,

$$X_t = x + c_0 t + \int_0^t b_0(Y_s)ds + \sqrt{a_0}\, B_t + \gamma_0\varphi_t, \tag{28}$$

$$Y_t = g(X_T) + \int_t^T f_0(Y_s)ds + \int_t^T h_0(Y_s)d\varphi_s + M_t - M_T, \tag{29}$$

where

$$\begin{aligned} b_0(y) &= \int_{\mathbb{T}^d}\Big(I + \nabla\hat{b}\Big)a(x)\frac{\partial^2\hat{e}}{\partial x\partial y}(x,y)m(dx), \qquad c_0 = \int_{\mathbb{T}^d}\Big(I + \nabla\hat{b}\Big)c(x)m(dx), \\ f_0(y) &= \int_{\mathbb{T}^d}\left(f + \left[\left\langle\frac{\partial\hat{e}}{\partial x}, c(x)\right\rangle - \left(\frac{\partial\hat{e}}{\partial y} \times e\right) + \frac{\partial^2\hat{e}}{\partial x\partial y}a(x)\left(\frac{\partial\hat{e}}{\partial x}\right)^*\right]\right)(x,y)m(dx), \\ h_0 &= \int_{\mathbb{T}^{d-1}}\left(h(.,y) + \left\langle\frac{\partial\hat{e}}{\partial x}(.,y), \gamma\right\rangle\right)(x)m_0(dx) \\ a_0 &= \int_{\mathbb{T}^d}\Big(I + \nabla\hat{b}\Big)(x)a(x)\Big(I + \nabla\hat{b}\Big)^*(x)m(dx), \end{aligned}$$

to state the result that follows.

Theorem 3.2. *Under (H),(16) and the conditions (2) to (8); we have*

$$Y_0^\varepsilon \to Y_0 \ \text{ in } \ \mathbb{R}.$$

Proof. By following the same techniques as in [4]. ■

4. Main Result

For each $(t,x) \in \mathbb{R}_+ \times \overline{\boldsymbol{D}}$, the solution of (1) is of the form

$$u^\varepsilon(t,x) := Y_0^\varepsilon, \tag{30}$$

where Y^ε denotes the solution of the BSDE considered in the previous section. Now, let us consider the following homogenized system :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = L\, u(t,x) + f_0\,(u(t,x)) \, + b_0(u(t,x)) \bigtriangledown u(t,x), \ x \in D, \\ \Gamma_0\, u(t,x) + h_0(u(t,x) = 0, \ x \in \partial D, \ t \geq 0, \\ u(0,x) = g(x), \ x \in D, \end{cases} \tag{31}$$

where

$$L = \frac{1}{2}\sum_{i,j=1}^{d} (a_0)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} c_0^i \frac{\partial}{\partial x_i}.$$

We shall assume w.l.o.g. that the orthogonal basis of $\mathbb{R}^d$ has been chosen in such a way so as the matrix a_0 is of the form

$$a_0 = \begin{pmatrix} a_0' & 0 \\ 0 & 0 \end{pmatrix},$$

where a_0' is a $d' \times d'$ positive definite matrix, with $d' \leq d$.
We set $\mathbb{R}^d = E_{d'} \oplus E_{d-d'}$, where $E_{d'}$ is the subspace of $\mathbb{R}^d$ of dimension d' generated by the vectors e_i, $i := 1,\ldots,d'$ after a new arrangement of the basis vectors of $\mathbb{R}^d$, in order to obtain the desired form of a_0.
Define the space

$$\mathbb{H}_{a_0}(\boldsymbol{D}) = \Big\{ v \in L^2(\overline{\boldsymbol{D}}) : \ \text{ such that } \ \sqrt{a_0} \bullet \nabla v \in (L^2(\overline{\boldsymbol{D}}))^d \ \text{ and } \ v \mid_{\partial \boldsymbol{D}} = 0 \Big\},$$

which is associated with the norm :

$$\| v \|_{\mathbb{H}_{a_0}(\boldsymbol{D})} = \Big(\| v \|^2_{L^2(\overline{\boldsymbol{D}})} + \| \sqrt{a_0}\, \nabla v \|^2_{(L^2(\overline{\boldsymbol{D}}))^d} \Big)^{1/2}.$$

By our assumptions , we have the *a priori* estimates

$$\| f_0(v) \|_{L^2(\overline{\boldsymbol{D}})} + \| b_0(v) \nabla v \|_{L^2(\overline{\boldsymbol{D}})} \leq C\Big(1 + \| v \|_{\mathbb{H}_{a_0}(\boldsymbol{D})} \Big).$$

Thus, we can state the main result of this work.

Theorem 4.1. *Under (H),(16) and the conditions (2) to (8), the system (31) has a unique solution u in* $L^2([0,T], H_0^1(\overline{\boldsymbol{D}}))$*, such that for all* $1 \leq k \leq d$,

$$\langle a_0 \nabla u_k, \nabla u_k \rangle \in L^1([0,T] \times \overline{\boldsymbol{D}}), \quad \text{with } u_k = \frac{\partial u}{\partial x_k}.$$

Moreover

$$u \in C(\mathbb{R}_+ \times \overline{\boldsymbol{D}}),$$

and we have for all $t \geq 0$ *, and for all* $x \in R^d$,

$$u^\varepsilon(t,x) \to u(t,x) \text{ , when } \varepsilon \to 0,$$

where $u^\varepsilon(t,x)$ *is the solution of the PDE system (1).*

Proof. We adopt similar tools as in [5].

∗ *Step 1*:

We first assume that the matrix a_0 is elliptic and we look for a solution

$$u \in L^2((0,T); H_0^1(\overline{\boldsymbol{D}})) \bigcap C([0,T]; L^2(\overline{\boldsymbol{D}})).$$

Then we prove the existence and uniqueness of the solution of the PDE. Set $\mathbb{F}(\boldsymbol{D}) = L^2((0,T); H_0^1(\overline{\boldsymbol{D}}))$ and consider the map :
$\Psi : \mathbb{F}(\boldsymbol{D}) \longrightarrow \mathbb{F}(\boldsymbol{D})$.
Let us show that Ψ is a contraction. For $\bar{v} \in \mathbb{F}$, consider $\bar{u} = \Psi(\bar{v})$ where $\bar{u} = u - u'$ and $\bar{v} = v - v'$. Denote by ν the ellipticity constant of a_0. For any $\alpha > 0$, we have

$$\begin{aligned}
&\frac{1}{2} e^{-\alpha t} \|\bar{u}_t\|^2_{L^2(\overline{\boldsymbol{D}})} + \nu \int_0^t e^{-\alpha s} \|\nabla \bar{u}_s\|^2_{(L^2(\overline{\boldsymbol{D}}))^d} ds \\
&\leq -\frac{\alpha}{2} \int_0^t e^{-\alpha s} \|\bar{u}_s\|^2_{L^2(\overline{\boldsymbol{D}})} ds + \int_0^t e^{-\alpha s} \langle h_0(v_s) - h_0(v'_s), \bar{u}_s \rangle_{L^2(\overline{\boldsymbol{D}})} d\varphi_s \\
&\quad + \int_0^t e^{-\alpha s} \langle f_0(v_s) - f_0(v'_s), \bar{u}_s \rangle_{L^2(\overline{\boldsymbol{D}})} ds.
\end{aligned}$$

With $\beta < 0$, remark that

$$\left\langle h_0(v) - h_0(v', \bar{u}) \right\rangle_{L^2(\overline{\boldsymbol{D}})} \leq \beta \|\bar{v}\|_{L^2(\overline{\boldsymbol{D}})} \|\bar{u}\|_{L^2(\overline{\boldsymbol{D}})},$$
$$\left\langle f_0(v) - f_0(v', \bar{u}) \right\rangle_{L^2(\overline{\boldsymbol{D}})} \leq \mu \|\bar{v}\|_{L^2(\overline{\boldsymbol{D}})} \|\bar{u}\|_{L^2(\overline{\boldsymbol{D}})}.$$

From this, it follows that

$$\begin{aligned}
&\frac{1}{2} e^{-\alpha t} \|\bar{u}_t\|^2_{L^2(\overline{\boldsymbol{D}})} + \nu \int_0^t e^{-\alpha s} \|\nabla \bar{u}_s\|^2_{(L^2(\overline{\boldsymbol{D}}))^d} ds \\
&\leq -\frac{\alpha}{2} \int_0^t e^{-\alpha s} \|\bar{u}_s\|^2_{L^2(\overline{\boldsymbol{D}})} ds + \mu \int_0^t e^{-\alpha s} \|\bar{v}_s\|_{L^2(\overline{\boldsymbol{D}})} \|\bar{u}_s\|_{L^2(\overline{\boldsymbol{D}})} ds.
\end{aligned}$$

By the fact that :
$(\nu X - \mu Y)^2 \geq 0 \Rightarrow XY \leq \frac{\nu}{2\mu} X^2 + \frac{\mu}{2\nu} Y^2,$
we have

$$\frac{1}{2}e^{-\alpha t}\|\overline{u}_t\|^2_{L^2(\overline{\boldsymbol{D}})} + \nu\int_0^t e^{-\alpha s}\|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}ds + \frac{\alpha}{2}\int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds$$
$$\leq \frac{\nu}{2}\int_0^t e^{-\alpha s}\Big(\|\overline{v}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \underbrace{\|\nabla\overline{v}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}}_{\text{we add this term}}\Big)ds$$
$$+\frac{\mu^2}{2\nu}\int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds.$$

Thereby,

$$\nu\int_0^t e^{-\alpha s}\|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}})^d}ds + \left(\frac{\alpha}{2} - \frac{\mu^2}{2\nu}\right)\int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds$$
$$\leq \frac{\nu}{2}\int_0^t e^{-\alpha s}\Big(\|\overline{v}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\nabla\overline{v}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}\Big)ds.$$

Choose $\alpha = 2\nu + \frac{\mu^2}{2\nu}$, then we have

$$\int_0^t e^{-\alpha s}\|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}ds + \int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds$$
$$\leq \frac{1}{2}\int_0^t e^{-\alpha s}\Big(\|\overline{v}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\nabla\overline{v}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}\Big)ds.$$

There follows that Ψ is a contraction on $\mathbb{F}(\boldsymbol{D})$ with the norm:

$$\|\overline{u}\|_\alpha = \left(\int_0^t e^{-\alpha s}\Big(\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}\Big)ds\right)^{\frac{1}{2}}.$$

$\ast$ *Step 2 :*

Consider the perturbed matrix $A^n = a_0 + \frac{1}{n}\boldsymbol{I}_d$, where a_0 can be degenerate. Let u^n be the unique solution of (31) after substituting a_0 to A^n. Multiply the equations of (31) by u^n to obtain

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{dt}\int_{\boldsymbol{D}}|u^n(t,x)|^2dx + \tfrac{1}{2}\int_{\boldsymbol{D}}\langle A^n\nabla u^n(t,x),\nabla u^n(t,x)\rangle dx\\
&= \int_{\boldsymbol{D}} b_0(u^n(t,x)\cdot\nabla(u^n(t,x))u^n(t,x)\,dx + \int_{\boldsymbol{D}} f_0(u^n(t,x))\,u^n(t,x)\,dx\\
&+\int_{\partial\boldsymbol{D}}\Gamma_0(u^n(t,x))u^n(t,x)d\varsigma + \int_{\partial\boldsymbol{D}} h_0(u^n(t,x))u^n(t,x)d\varsigma\\
&+\tfrac{1}{2}\int_{\partial\boldsymbol{D}} c_0\cdot\frac{\partial}{\partial x}(u^n(t,x)^2)dx,
\end{aligned}$$

where ς is the $(d-1)$-dimensional volume element on $\partial\boldsymbol{D}$.
First we note that

$$\int_{\boldsymbol{D}} c_0\cdot\frac{\partial}{\partial x}(u^n(t,x)^2)dx = 0 \quad \text{a.e.,} \quad \text{and} \quad \int_{\partial\boldsymbol{D}}\Gamma_0(u^n(t,x))u^n(t,x)d\varsigma = 0.$$

Second, by the boundedness of σ and $\frac{\partial^2 \hat{e}}{\partial x \partial y}$, one can easily show that

$$\left| \int_{\boldsymbol{D}} b_0(u^n(t,x)) \cdot \nabla(u^n(t,x)) u^n(t,x) dx \right| \leq K \int_{\boldsymbol{D}} \| \sqrt{a_0} \nabla u^n(t,x) \| |u^n(t,x)| dx.$$

Thus

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\boldsymbol{D}} |u^n(t,x)|^2 dx + \frac{1}{2} \int_{\boldsymbol{D}} \langle A^n \nabla u^n(t,x), \nabla u^n(t,x) \rangle dx & \\ \leq \mu \int_{\boldsymbol{D}} |u^n(t,x)|^2 dx + K \int_{\boldsymbol{D}} \left\| \sqrt{A^n} \nabla u^n(t,x) \right\| |u^n(t,x)| dx & \\ \leq \left(\mu + \frac{K^2}{2\delta} \right) \int_{\boldsymbol{D}} |u^n(t,x)|^2 dx + \frac{\delta}{2} \int_{\boldsymbol{D}} \langle A^n \nabla u^n, \nabla u^n \rangle (t,x) dx. & \end{aligned}$$

Choose $\delta = \frac{1}{2}$, to deduce by Gronwall's lemma,

$$\int_{\boldsymbol{D}} |u^n(t,x)|^2 dx \leq K' \mathrm{e}^{K't} \qquad \text{and} \qquad \int_0^T \int_{\boldsymbol{D}} \langle A \nabla u^n(t,x), \nabla u^n(t,x) \rangle dx dt \leq k(T).$$

Now we differentiate the equations of (31) for u^n with respect to x_k. Then $u_k^n = \frac{\partial u^n}{\partial x_k}$ satisfies

$$\begin{cases} \frac{\partial u_k^n}{\partial t}(t,x) = \frac{1}{2} \sum_{i,j=1}^{d} A_{ij}^n \frac{\partial^2 u_k^n}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} c_0^i \frac{\partial u_k^n}{\partial x_i}(t,x) \\ \qquad + f_0'(u^n(t,x)) u_k^n(t,x) + b_0'(u^n(t,x)) u_k^n(t,x) \nabla u^n(t,x) \\ \qquad + b_0(u^n(t,x)) \nabla u_k^n(t,x), \qquad x \in \boldsymbol{D}, \\ \Gamma_0[u_k^n(t,x)] + h_0'(u^n(t,x)) u_k^n(t,x) = 0, \qquad x \in \partial \boldsymbol{D}, \\ u_k^n(0,x) = \frac{\partial g}{\partial x}(x). \end{cases} \tag{32}$$

Note that

$$\int_{\boldsymbol{D}} c_0 \cdot \frac{\partial}{\partial x} (u_k^n(t,x)^2) dx = 0 \quad \text{a.e.,} \quad \text{and} \quad \int_{\partial \boldsymbol{D}} \Gamma_0[u_k^n(t,x)] u_k^n(t,x) d\varsigma = 0.$$

From this, by multiplying the equations in (32) by u_k^n, we have

$$\begin{aligned} & \tfrac{1}{2} \frac{d}{dt} \int_{\boldsymbol{D}} |u_k^n(t,x)|^2 dx + \tfrac{1}{2} \int_{\boldsymbol{D}} \langle A^n \nabla u_k^n(t,x), \nabla u_k^n(t,x) \rangle dx \\ & = \int_{\boldsymbol{D}} [b_0'(u^n(t,x)) \nabla u^n(t,x) (u_k^n(t,x))^2 dx + b_0(u^n(t,x)) \nabla u_k^n(t,x) u_k^n(t,x)] dx \\ & \quad + \int_{\boldsymbol{D}} f_0{}'(u^n(t,x))\, (u_k^n(t,x))^2 dx + \int_{\partial \boldsymbol{D}} h_0{}'(u^n(t,x)) (u_k^n(t,x))^2 d\varsigma. \end{aligned}$$

Remark that

- $\langle b_0'(u^n(t,x)) \nabla u^n(t,x), (u_k^n(t,x))^2 \rangle = -b_0'(u^n(t,x))\, u^n(t,x) \nabla u_k^n(t,x) u_k^n(t,x)$,
- $\langle h_0'(u^n(t,x)), (u_k^n(t,x))^2 \rangle \leq \beta |(u^n(t,x)| |(u_k^n(t,x)|^2 \leq \beta' \, |(u_k^n(t,x)|^2, \ \beta' < 0$,
- $\left\langle f_0'(u^n(t,x)), (u_k^n(t,x))^2 \right\rangle \leq \mu |(u^n(t,x)| |(u_k^n(t,x)|^2 \leq \mu' \, |(u_k^n(t,x)|^2$.

Thereafter,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{D}}|u_k^n(t,x)|^2dx + \frac{1}{2}\int_{\boldsymbol{D}}\langle A^n\nabla u_k^n(t,x),\nabla u_k^n(t,x)\rangle dx \\ \leq \mu'\int_{\boldsymbol{D}}|u_k^n(t,x)|^2dx + K\int_{\boldsymbol{D}}\left\| \sqrt{A^n}\,\nabla u_k^n(t,x)\right\| |u_k^n(t,x)|dx \\ \leq \left(\mu' + \frac{K^2}{2\delta}\right)\int_{\boldsymbol{D}}|u_k^n(t,x)|^2dx + \frac{\delta}{2}\int_{\boldsymbol{D}}\langle A^n\nabla u_k^n,\nabla u_k^n\rangle(t,x)dx.\end{aligned}$$

By an appropriate choice of δ, and using Gronwall's lemma, we can write
$\int_{\boldsymbol{D}}|u_k^n(t,x)|^2dx \leq K\mathrm{e}^{Kt}$.
Hence, we have proved that u^n is bounded in $L^\infty([0,T];H_0^1(\overline{\boldsymbol{D}}))$; also that each u_k^n is bounded in $L^2([0,T];\mathbb{H}_{a_0}(\boldsymbol{D}))$.

∗ *Step 3:*
Let us show that u^n is a Cauchy sequence in $L^2([0,T];\mathbb{H}_{a_0}(\boldsymbol{D}))$,

$$\begin{aligned}\frac{\partial(u^n-u^m)}{\partial t}(t,x) = {} & \frac{1}{2}\sum_{i,j=1}^{d}(a_0)_{ij}\frac{\partial^2(u^n-u^m)}{\partial x_i\partial x_j}(t,x) + \frac{1}{2n}\sum_{i,j=1}^{d}\frac{\partial^2 u^n}{\partial x_i\partial x_j}(t,x) \\ & - \frac{1}{2m}\sum_{i,j=1}^{d}\frac{\partial^2 u^m}{\partial x_i\partial x_j}(t,x) + \sum_{i=1}^{d}c_0^i\frac{\partial(u^n-u^m)}{\partial x_i}(t,x) \\ & + f_0(u^n(t,x)) - f_0(u^m(t,x)) + b_0(u^n(t,x))\nabla u^n(t,x) \\ & - b_0(u^m(t,x))\nabla u^m(t,x) + \Gamma_0[(u^n-u^m)(t,x)] \\ & + h_0(u^n(t,x)) - h_0(u^m(t,x)).\end{aligned}$$

Then by multiplying this equation by (u^n-u^m), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\ \|u^n-u^m\|^2(t) + \frac{1}{2}\int_{\boldsymbol{D}}\langle a_0\bigtriangledown(u^n-u^m),\bigtriangledown(u^n-u^m)\rangle(t,x)dx \\ &+ \frac{1}{2}\int_{\boldsymbol{D}}\Big\langle\Big(\frac{1}{n}\bigtriangledown u^n - \frac{1}{m}\bigtriangledown u^m\Big),\bigtriangledown(u^n-u^m)\Big\rangle(t,x)dx \\ = {} & \frac{1}{2}\int_{\boldsymbol{D}}\ \sum_{i=1}^{d}c_0^i\frac{\partial[(u^n-u^m)^2]}{\partial x_i}(t,x)dx + \int_{\boldsymbol{D}}\Big\langle f_0(u^n)-f_0(u^m),(u^n-u^m)\Big\rangle(t,x)dx \\ &+ \int_{\boldsymbol{D}}\Big\langle\ b_0(u^n)\bigtriangledown u^n - b_0(u^m)\bigtriangledown u^m,(u^n-u^m)\Big\rangle(t,x)dx \\ &+ \int_{\partial\boldsymbol{D}}\{\Gamma_0[(u^n-u^m)](u^n-u^m) + \langle h_0(u^n)-h_0(u^m),(u^n-u^m)\rangle\}\ (t,x)d\varsigma.\end{aligned}$$

Observe that

$$\int_{\boldsymbol{D}}c_0\cdot\frac{\partial}{\partial x}\big[(u^n-u^m)^2\big]dx = 0 \quad \text{a.e., and} \quad \int_{\partial\boldsymbol{D}}\Gamma_0[(u^n-u^m)](u^n-u^m)(t,x)d\varsigma = 0,$$

to integrate with respect to t, and obtain

$$\frac{1}{2}\frac{d}{dt}\ \|u^n-u^m\|^2(t)+\frac{1}{2}\int_0^t\int_{\boldsymbol{D}}\langle a_0\nabla(u^n-u^m),\nabla(u^n-u^m)\rangle(s,x)dxds$$
$$+\frac{1}{2}\int_0^t\int_{\boldsymbol{D}}\Big\langle\Big(\frac{1}{n}\nabla u^n-\frac{1}{m}\nabla u^m\Big),\nabla(u^n-u^m)\Big\rangle(s,x)dxds$$
$$=\frac{1}{2}\int_0^t\int_{\boldsymbol{D}}\ \sum_{i=1}^d c_0^i\frac{\partial[(u^n-u^m)^2]}{\partial x_i}(t,x)dx+\int_{\boldsymbol{D}}\Big\langle f_0(u^n)-f_0(u^m),(u^n-u^m)\Big\rangle(s,x)dxds$$
$$+\int_0^t\int_{\boldsymbol{D}}\Big\langle b_0(u^n)\nabla u^n-b_0(u^m)\nabla u^m,(u^n-u^m)\Big\rangle(s,x)dxds$$
$$+\int_0^t\int_{\partial\boldsymbol{D}}\{\Gamma_0[(u^n-u^m)](u^n-u^m)+\langle h_0(u^n)-h_0(u^m),(u^n-u^m)\rangle\}\ (s,x)d\varsigma d\varphi_s.$$

Since ∇u^n and ∇u^m are bounded in $L^2([0,T];\overline{\boldsymbol{D}})^d$,

$$\frac{1}{2}\int_0^T\int_{\boldsymbol{D}}\Big\langle\Big(\frac{1}{n}\nabla u^n-\frac{1}{m}\nabla u^m\Big),\nabla(u^n-u^m)\Big\rangle(t,x)dxdt,$$

tends to zero whenever n and m tend to infinity.
For $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that for $n,m\geq N_\varepsilon$, all $\delta>0$

$$\frac{1}{2}\|u^n-u^m\|^2(t)+\frac{1-\delta}{2}\int_0^t\int_{\boldsymbol{D}}\langle a_0(\nabla u^n-\nabla u^m),(\nabla u^n-\nabla u^m)\rangle(s,x)dxds$$
$$\leq\varepsilon+\Big(\mu'+\frac{K^2}{2\delta}\Big)\int_0^t\int_{\boldsymbol{D}}\|u^n-u^m\|^2(s,x)dxds.$$

Hence by choosing $\delta=\frac{1}{2}$ and exploiting Gronwall's lemma, we have

$$\frac{1}{2}\|u^n-u^m\|^2(t)+\frac{1}{4}\int_0^t\int_{\boldsymbol{D}}\langle a_0(\nabla u^n-\nabla u^m),(\nabla u^n-\nabla u^m)\rangle(s,x)dxds$$
$$\leq\varepsilon e^{Kt},\quad\forall n,m\geq N_\varepsilon,\quad t\in[0,T].$$

There follows that u^n is a Cauchy sequence in $L^2([0,T];\mathbb{H}_{a_0}(\boldsymbol{D}))$, and there exists $u\in L^2([0,T];\mathbb{H}_{a_0}(\boldsymbol{D}))$ such that
$u^n\to u$ in $L^2([0,T];\mathbb{H}_{a_0}(\boldsymbol{D}))$.
Moreover since

$$\int_0^T\int_{\boldsymbol{D}}\langle f_0(u^n)-f_0(u),(u^n-u)\rangle(t,x)dxdt$$
$$+\int_0^T\int_{\boldsymbol{D}}\langle b_0(u^n)\nabla u^n-b_0(u)\nabla u,(u^n-u)\rangle(t,x)dxdt$$
$$\leq K\int_0^T\int_{\boldsymbol{D}}\Big\{\|u^n-u\|^2(t,x)+\langle a_0(\nabla u^n-\nabla u),(\nabla u^n-\nabla u)\rangle(s,x)\Big\}dxdt,$$

and

$$\int_0^T \int_{\partial \boldsymbol{D}} |\langle h_0(u^n) - h_0(u), (u^n - u)\rangle| d\varsigma d\varphi_t \leq K' \int_0^T \int_{\partial \boldsymbol{D}} \|u^n - u\|^2 d\varsigma d\varphi_t\ ,$$

then

$$f_0(u^n) + b_0(u^n)\nabla u^n \longrightarrow f_0(u) + b_0(u)\nabla u \quad \text{in} \quad L^2([0,T];\mathbb{H}_{a_0}(\boldsymbol{D})).$$
$$h_0(u^n) \longrightarrow h_0(u) \quad \text{in} \quad L^2([0,T];\partial \boldsymbol{D}).$$

Moreover the sequence $\{u^n\}$is bounded in $\mathbb{F}(\boldsymbol{D})$, hence $u$is in $\mathbb{F}(\boldsymbol{D})$.
By similar arguments, one can easily illustrate the uniqueness of the solution $u$in $\mathbb{F}(\boldsymbol{D})$. ■

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