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BSDENMs: Enlargement of Filtration and Insider Trading

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Abstract. T*he aim of this paper is to study backward stochastic differential equations driven by general normal martingales* $(M_t)_{t\in[0,T]}$ *with deterministic bracket* $\langle M,M\rangle_t$ *(BSDENM), having the chaos representation property. We prove the existence and uniqueness of the pertaining solution under the stochastic Lipschitz condition. In this work, the backward stochastic differential equations, (BSDENM) are wealth equations, and we use Jacod's method of enlargement of filtration to model the asymmetrical information. We also compare the strategies of an insider trader and a non insider one.*

Key words: Filtration, Asymmetrical Information, Insider Trading, Normal Martingale, Azéma Martingale.

AMS Subject Classifications: 60F44, 60H05, 90A09, 60H30

1. Introduction

The theory of backward stochastic differential equations (BSDE) was developed in 1990 by Pardoux and Peng [26]. They proved the existence and uniqueness of adapted processes (Y,Z) solution of the following equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T,$$

where W is a Brownian motion and the generator f is uniformly Lipschitz. In this paper, we do not refer to a Brownian filtration and we will consider a BSDE driven by general normal martingales (BSDENM). Moreover, since a lot of problems do not satisfy the uniformly Lipschitz condition, then we shall be interested in a non deterministic Lipschitz condition. For example, we shall entertain the pricing problem which is equivalent to solving the following Linear BSDENM:

$$\begin{cases} dY_t &= (r_tY_t + Z_t\theta_t)dt + Z_t dM_t, \qquad 0 \le t \le T, \\ Y_T &= \xi, \end{cases}$$

where ξ is the contingent claim to price and hedge, r is the short rate of the interest and θ is a real valued process.

Replicating hedging strategies in complete markets [22] for European options are explicitly computed using the Clark-Ocone formula [9]. However explicit computations are in general difficult to perform via this formula. For markets driven by Brownian motion, a proof of the classical Black-Scholes formula via the Clark-Ocone formula can be found in [23], Ch. 5, p. 13. This method has recently been extended to markets driven by a Poisson process in [5]. Brownian motion and the compensated Poisson process share the important chaos representation property, which is capital for market completeness. In this paper we consider a martingales $(M_t)_{t\in[0,T]}$ having the chaos representation property (in relation with market completeness), i.e. every square-integrable functional, measurable with respect to the filtration generated by $(M_t)_{t\in[0,T]}$, can be expanded into a series of multiple stochastic integrals of deterministic functions with respect to $(M_t)_{t\in[0,T]}$.

Furthermore, the present paper focuses more on option hedging problems. Here, we extend the results given for hedging in market models driven by Brownian motion and the Poisson process to the one driven by a martingale $(M_t)_{t\in[0,T]}$. We have chosen to study wealth equations as BSDENM with deterministic bracket $\langle M, M\rangle_t$, having the chaos representation property. The traders have different information on the market, so we say that they are in an asymmetric information situation. Indeed, we do this by considering a financial market that is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ (public information). While the ordinary trader makes its decisions based on the information flow $(\mathcal{F}_t)_{t\in[0,T]}$, the insider processes from the beginning extra information about the outcome of some random variable G taking values in a Polish space $(\mathbf{X}, \mathcal{X})$. The insider's information flow is therefore described by the enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]} = (\mathcal{F}_t \vee \sigma(G))_{t\in[0,T]}$. The problem of the enlargement of a filtration consists of the following three important issues. First we recall conditions on the random variable G such that every $\mathbf{F}$–semimartingale becomes a $\mathbf{G}$–semimartingale, second we transfer the chaos representation property from $\mathbf{F}$ to $\mathbf{G}$, and third we get, for a version of the stochastic integrals under $\mathbf{F}$, the common version under $\mathbf{G}$. The purpose of this analysis is to compare the strategies of an insider trader and a non insider one, under different models.

In the second section we give an example of the financial market driven by the so called Azéma martingale. The third section is devoted to the existence and uniqueness of the solution of the BSDENM under the stochastic Lipschitz condition then we give the pertaining comparison theorem. Section four deals with the problem of anticipation on financial markets. We study wealth equations as BSDENM, and we use the J. Jacod's method of enlargement of filtration (see [19]) to model the asymmetrical information used by the insider trader on the financial market. This section is finished with comparing the strategies of the insider trader and a non insider one.

2. Preliminary Notes

Let $(M_t)_{t\le T}$ be a *normal martingale* on the space $\Omega = \mathcal{C}_d([0,T];\mathbb{R})$ i.e., a martingale with cádlág paths such that $(M_t^2 - t)_{t\le T}$ is also a martingale, both with respect to a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\le T}$ which is right continuous and such that $\mathcal{F}_0$ contains all $\mathcal{P}$–null sets; it is assumed throughout that the filtration $\mathbf{F}$ is generated by M.

We assume that $(M_t)_{t\le T}$ is solution of the structural equation

$$d[M,M]_t = dt + \beta M_{t-} dM_t, \quad 0 \leq t \leq T.$$

Recall that this equation has a unique solution, called the Azéma martingale, for all $\beta \in \mathbb{R}$ (see [16]). Moreover, if $\beta \in [-2,0]$, then M has the predictable representation property.

2.1. **Market model and probability measure**

Let $\mu : [0,T] \to \mathbb{R}$, $\sigma : [0,T] \to \mathbb{R}_*^+$ be bounded functions and $r : [0,T] \to \mathbb{R}_*^+$ which models a risk-less asset. Let $(\psi)_t)_{t\in[0,T]}$ be defined as

$$\psi_t = \frac{r_t - \mu_t}{\sigma_t}.$$

Assume that $1 + \phi_t \psi_t > 0, \ \forall t \in [0,T]$ and denotes by $\mathbf{Q}$ the probability defined by

$$\frac{d\mathbf{Q}}{d\mathbf{P}} / \mathcal{F}_t = \zeta_t,$$

where ζ is solution of the "Doléans-Dade" equation

$$\zeta_t = 1 + \int_0^t \zeta_{s^-} \psi_s dM_s.$$

If we assume

$$N_t = M_t - \int_0^t \psi_s ds,$$

then the process N is a martingale under $\mathbf{Q}$ and

$$\begin{cases} d\langle N,N\rangle_t &= (1 + \beta M_t \psi_t) dt \\ d[N,N]_t &= d\langle N,N\rangle_t + \beta M_{t-} dN_t. \end{cases}$$

Consider next a financial market where there are a non risky asset S^0 and a risky asset S^1 which are traded continuously

$$\begin{aligned} dS_t^0 &= r_t S_t^0 dt, \qquad & S_0^0 &= 1 \\ dS_t &= \mu_t S_t dt + \sigma_t S_{t^-} dN_t \qquad & S_0 &= x. \end{aligned} \tag{1}$$

The equation (1) can be written as

$$dS_t = r_t S_t dt + \sigma_t S_{t-} dM_t.$$

Then, under $\mathbf{P}$, the actualized price $e^{-\int_0^t r_s ds} S_t$ is a martingale and the market is arbitrage free. Let $\pi = (\pi^0, \pi)$ be a portfolio for a small investor. The process π^0 and π denote the amount invested in the bond and the risky asset respectively. the portfolio π is called self financing if its value $(V_t)_{t\leq T}$ satisfies

$$dV_t = \pi_t^0 \frac{dS_t^0}{S_t^0} + \pi_t \frac{dS_t}{S_t} = r_t V_t dt + \pi_t \sigma_t dM_t.$$

Now, let us consider $V_T = \xi$ a non negative contingent claim at time T. The point is that we want to seek for the initial endowment V_0 for the investor allowing him to have a terminal wealth equal to ξ at the terminal time T in making the self financing investment π. Indeed, if $\xi \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P})$, the value V_0 is equal to Y_0 where (Y,Z) is solution of the linear BSDE

$$\begin{cases} dY_t &= r_t Y_t dt + Z_t dM_t, \qquad 0 \le t \le T, \\ Y_T &= \xi. \end{cases}$$

3. BSDEs Driven by a Normal Martingale

In what follows, we consider a large family of *normal martingales* $(M_t)_{t \le T}$ satisfying the two conditions:
- The predictable representation property,
- $d\langle M,M\rangle_t = \alpha_t^2 dt$, where α is a positive deterministic non vanishing square-integrable function.

This implies that there exists a predictable, M-integrable process ϕ such that the martingale M verifies the structural equation

$$d[M,M]_t = \alpha_t^2 dt + \phi_t dM_t. \tag{2}$$

Remark 3.1. The continuous part of $(M_t)_{t \le T}$ is given by $dM_t^c = \mathbf{I}_{(\phi_t=0)} dM_t$ and its eventual jump is given as $\Delta M_t = \phi_t, \ \forall t \le T$.

Let $\lambda > 0$ be a real and put $A_t = \int_0^t \alpha_s^2 ds$. We consider the data (ξ,f) defined as follows:
$(\mathcal{H}.1)$ ξ is an $\mathcal{F}_T$–measurable random variable such that $\mathbf{E} e^{\lambda A_T}|\xi|^2 < \infty$.
$(\mathcal{H}.2)$ a function f, which is map $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:
1. $\forall (y,z) \in \mathbb{R}^2 : \ (\omega,t) \longrightarrow f(\omega,t,y,z)$ is $\mathcal{F}_t$ - progressively measurable.
2. $\mathbf{E} \int_0^T e^{\lambda A_t} \dfrac{|f(t,0,0)|^2}{\alpha_t^2} dt < \infty$.
3. There exist two non-negative $\mathcal{F}_t$–adapted processes $(u_t)_{t \le T}$ and $(v_t)_{t \le T}$ such that: for all $t \in [0,T]$ and $y,y',z,z' \in \mathbb{R}$ $|f(t,y,z) - f(t,y',z')| \le u_t|y-y'| + v_t|z-z'|$.
We suppose also that: $\exists \varepsilon > 0, \quad \alpha_t^2 \ge u_t + |v_t|^2 \ge \varepsilon$.

Definition 3.1. The continuous A pair (Y,Z) is called a solution of BSDENM with data (ξ,f) if
- Y is adapted and cádlág process such that $\mathbf{E} \sup_{0 \le t \le T} e^{\lambda A_t}|Y_t|^2 < \infty$.
- Z is a predictable process such that $\mathbf{E} \int_0^T e^{\lambda t}|Z_t|^2 dA_t < \infty$.
- The process (Y,Z) verifies the following BSDE

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dM_s. \tag{3}$$

Remark 3.2.
- If $\alpha = 1$ and $\phi = 0$, then M will be a standard Brownian motion. In this case, existence and uniqueness for the solution of (3) is full study by Pardoux-Peng [26] when the coefficient is uniformly Lipschitz and by Bender-Kohlmann [8] in the stochastic Lipschitz condition.
- If $\alpha = 1$ and $\phi_t = M_{t-}$, then M is a Azéma martingale. The associated BSDE is introduced by Rainer [29]. The author discus also the BSDE with an asymmetric martingale (i.e. $(\phi_t = M_{t-}^+)$.
- The case when $\phi = 0$ and α predictable is treated in a general formulation by El-Karoui and Huang, [12].

In this work, we are interested on the existence and uniqueness for solution of BSDENM (3).

3.1. **Uniqueness**

Proposition 3.1. *Let* (Y,Z) *and* (Y',Z') *be two solutions of BSDENM (3) associated respectively with data* (ξ,f) *and* (ξ',f'). *Then,*

$$\mathbf{E}\sup_{0\le t\le T} e^{\lambda A_t}|Y_t - Y'_t|^2 + \mathbf{E}\int_0^T e^{\lambda A_s}[|Y_s - Y'_s|^2 + |Z_s - Z'_s|^2]dA_s$$
$$\le C(\lambda,T)\left\{\mathbf{E}e^{\lambda A_T}|\xi - \xi'^2 + \mathbf{E}\int_0^T e^{\lambda A_s}\frac{|f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)|^2}{\alpha_t^2}ds\right\}.$$

Proof. Denote by $\delta\eta = \eta - \eta'$ for $\eta = \xi, Y, Z$. From Itô's formula and the structural equation (2) , we have

$$e^{\lambda A_T}|\delta\xi|^2 = e^{\lambda A_t}|\delta Y_t|^2 + \lambda\int_t^T e^{\lambda A_s}|\delta Y_s|^2 dA_s$$
$$- 2\int_t^T e^{\lambda A_s}\delta Y_s[f(s,Y_s,Z_s) - f'(s,Y'_s,Z'_s)]ds + 2\int_t^T e^{\lambda A_s}\delta Y_s\delta Z_s dM_s$$
$$+ \int_t^T e^{\lambda A_s}|\delta Z_s|^2\alpha_s^2 ds + \int_t^T e^{\lambda A_s}|\delta Z_s|^2\phi_s dM_s.$$

By using Young's inequality and taking expectation, we can write in view of hypothesis $(\mathcal{H}.2)(2)$

$$\mathbf{E}e^{\lambda A_t}|\delta Y_t|^2 + \mathbf{E}\int_t^T e^{\lambda A_s}[\lambda|\delta Y_s|^2 + |\delta Z_s|^2]dA_s$$
$$\le \mathbf{E}e^{\lambda A_T}|\delta\xi|^2 + \frac{2}{\gamma}\mathbf{E}\int_t^T e^{\lambda A_s}\frac{|f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)|^2}{\alpha_s^2}]ds$$
$$+ \mathbf{E}\int_t^T e^{\lambda A_s}[2(\gamma+1)|\delta Y_s|^2 + \frac{1}{2}|\delta Z_s|^2]dA_s$$

As usual, the inequality stands for any positive γ. So if we choose $\lambda > \frac{3}{2}$ and $\gamma = \frac{1}{4}(2\lambda - 3)$. We conclude by Gronwall's Lemma that

$$\mathbf{E}e^{\lambda A_t}|\delta Y_t|^2 + \frac{1}{2}\mathbf{E}\int_0^T e^{\lambda A_s}[|\delta Y_s|^2 + |\delta Z_s|^2]dA_s$$
$$\le \mathbf{E}e^{\lambda A_T}|\delta\xi|^2 + C(\lambda,T)\mathbf{E}\int_0^T e^{\lambda A_s}\frac{|f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)|^2}{\alpha_s^2}]ds.$$

Then the result follows by applying Doob's inequality. ■

We deduce immediately from the last proposition the next result.

Corollary 3.1. *Under the assumptions* $(H.1)$ *and* $(H.2)$, *the BSDENM (3) has at most one solution.* ■

3.2. **Existence**

Theorem 3.1. *We suppose that the coefficient f does not depends on* (y,z). *Let* $f(s,y,z) = g(s)$, *the*

BSDENM (3) *with data* (ξ, g) *has a unique solution.*

Proof. First, using Schwarz inequality and assumptions $(\mathcal{H}.1)$ and $(\mathcal{H}.2)(1)$, one has

$$\mathbf{E}\left|\xi+\int_0^T g(s)ds\right|^2 \le 2\left\{\mathbf{E}e^{\lambda A_T}|\xi|^2+\mathbf{E}\left|\int_0^T e^{\frac{1}{2}\lambda A_t}\frac{g(t)}{\alpha_t}.e^{-\frac{1}{2}\lambda A_t}\alpha_t dt\right|^2\right\}$$

$$\le 2\left\{\mathbf{E}e^{\lambda A_T}|\xi|^2+\tfrac{1}{\lambda}\mathbf{E}\int_0^T e^{\lambda A_t}\frac{|g(t)|^2}{\alpha_t^2}dt\right\}<\infty. \tag{4}$$

The martingale representation theorem imply that there exists a unique predictable process Z, verifying $\mathbf{E}\int_0^T |Z_s|^2\alpha_s^2 ds<\infty$, such that

$$\xi+\int_0^T g(s)ds=\mathbf{E}\left[\xi+\int_0^T g(s)ds\right]+\int_0^T Z_s dM_s.$$

Let $Y_t=\mathbf{E}[\xi+\int_t^T g(s)ds\,/\mathcal{F}_t]$. Then the process (Y,Z) verifies the BSDENM (3) with data (ξ, g). On the other hand, in the same way that (4), we can prove that

$$\mathbf{E}e^{\lambda A_t}|Y_t|^2\le 2\mathbf{E}\left[e^{\lambda A_T}|\xi|^2+\frac{1}{\lambda}\int_0^T e^{\lambda A_s}\frac{|g(s)|^2}{\alpha_s^2}ds\,/\mathcal{F}_t\right].$$

By Doob's maximal quadratic inequality, we deduce that $\mathbf{E}\sup_{0\le t\le T}e^{\lambda A_t}|Y_t|^2<\infty$. Then, the pair (Y,Z) is the unique solution of (3). ■

Theorem 3.2. *Assume that the assumptions* $(\mathcal{H}.1)$ *and* $(\mathcal{H}.2)$ *holds. Then the BSDENM (3) with data* (ξ, f) *has a unique solution.*

Proof. Define the sequence (Y^n, Z^n) as follows: $(Y^0, Z^0)=(0,0)$ and (Y^{n+1}, Z^{n+1}) is solution of the BSDENM

$$Y_t^{n+1}=\xi+\int_t^T f(s,Y_s^n,Z_s^n)ds-\int_t^T Z_s^{n+1}dM_s.$$

We shall prove that (Y^n, Z^n) is a Cauchy sequence for the Banach space provided with norm

$$\|(Y,Z)\|^2_{(A,\lambda)}=\mathbf{E}e^{\lambda A_t}|Y_t|^2+\mathbf{E}\int_0^T e^{\lambda A_s}[|Y_s|^2+|Z_s|^2]dA_s.$$

For $n\ge m\ge 1$, let put $\eta^{n,m}=\eta^n-\eta^m$ for $\eta=Y,Z$. Using the same arguments that in the proof of Proposition 3.1, one has

$$\mathbf{E}\sup_{0\le t\le T}e^{\lambda A_t}|Y_t^{n+1,m+1}|^2+\mathbf{E}\int_t^T e^{\lambda A_s}[(\lambda-\gamma)|Y_s^{n+1,m+1}|^2+|Z_s^{n+1,m+1}|^2]dA_s$$

$$\le\frac{C}{\gamma}\mathbf{E}\int_0^T e^{\lambda A_s}\frac{|f(s,Y_s^n,Z_s^n)-f(s,Y_s^m,Z_s^m)|^2}{\alpha_s^2}ds.$$

So, for any $\epsilon>0$ and by the hypothesis $(\mathcal{H}.2)(2)$,

$$\frac{|f(s,Y_s^n,Z_s^n)-f(s,Y_s^m,Z_s^m)|^2}{\alpha_s^2}\le\alpha_s^2|Y_s^{n,m}|^2+\frac{1}{\varepsilon}\alpha_s^2|Z_s^{n,m}|^2.$$

We deduce for $\gamma=\lambda+1$ and λ large to C that

$$\|(Y^{n+1,m+1}, Z^{n+1,m+1})\|^2_{(A,\lambda)} \leq \frac{1}{2}\|(Y^{n,m}, Z^{n,m})\|^2_{(A,\lambda)}.$$

By iterating the last inequality, we deduce that $\|(Y^{n,m}, Z^{n,m})\|_{(A,\lambda)} \leq \frac{C}{2^n}$, which imply that the sequence (Y^n, Z^n) is a Cauchy sequence and converges on (Y, Z) satisfying (3). ■

3.3. **Comparison theorem**

Now let us pass to the comparison theorem. We suppose additionally that $\alpha = 1$ and
$(\mathcal{H}.3)$ $1 - v_s|\phi_s| > 0,\ \forall\ s \leq T.$
Let (ξ^i, f^i) for $i = 1,2$ be two sets of data which satisfies the assumptions $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$. For $i = 1,2$, (Y^i, Z^i) denotes a solution of BSDENM (3) with data (ξ^i, f^i). Thus, we have also the following comparison theorem between the solutions of the NBSDE's.

Theorem 3.3. (Comparison theorem) *We suppose that* $\xi^1 \geq \xi^2$ *and* $f^1(t,y,z) \geq f^2(t,y,z)$, $\forall (y,z)$ *$dP \times dt$–a.s. Then,* $Y^1_t \geq Y^2_t$ *for all* $0 \leq t \leq T$*–a.s.*

Proof. Set $\delta Y = Y^1 - Y^2$, $\delta Z = Z^1 - Z^2$, $\delta\xi = \xi^1 - \xi^2$ and $\delta f_s = f^1(s, Y^2_s, Z^2_s) - f^2(s, Y^2_s, Z^2_s)$. We defines two processes $\mathcal{A}$ and $\mathcal{B}$ as follows

$$\mathcal{A}_s = \begin{cases} (\delta Y_s)^{-1}(f^1(s, Y^1_s, Z^1_s) - f^1(s, Y^2_s, Z^1_s)) & \text{if} \quad \delta Y_s \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{B}_s = \begin{cases} (\delta Z_s)^{-1}(f^1(s, Y^2_s, Z^1_s) - f^1(s, Y^2_s, Z^2_s)) & \text{if} \quad \delta Z_s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In view of the above notations,

$$\delta Y_t = \delta Y_s - \int_s^t [\delta Y_r \mathcal{A}_r + \delta Z_r \mathcal{B}_r + \delta f_r] dr + \int_s^t \delta Z_r dM_r.$$

Now, let $(\Gamma_{s,t})_{t \in [s,T]}$ be solution of the linear stochastic differential equation

$$\Gamma_{s,t} = 1 + \int_s^t \mathcal{A}_r \Gamma_{s,r} dr + \int_s^t \mathcal{B}_r \Gamma_{s,r-} dM_r$$

which verifies (see [28], pp 84)

$$\Gamma_{s,t} = \exp\left\{\int_s^t [\mathcal{A}_r + \frac{1}{2}|\mathcal{B}_r|^2] dr\right\} \prod_{s<r\leq t} (1 + \mathcal{B}_r \Delta M_r) \exp(-\mathcal{B}_r \Delta M_r).$$

It follows by Remark 3.1 and $(\mathcal{H}.3)$ that $\Gamma_{s,t} > 0$. On the other hand, by integration by part formula with δY and $\Gamma_{s,\cdot}$, we get, in view of (2),

$$\delta Y_t \Gamma_{s,t} = \delta Y_s - \int_s^t \delta f_r \Gamma_{s,r} dr + \int_s^t [\delta Y_r \mathcal{B}_r \Gamma_{s,r-} + \delta Z_r \Gamma_{s,r-}(1 + \mathcal{B}_r \phi_r)] dM_r.$$

In particular for $t = T$. We obtain by taking the conditional expectation that

$$\delta Y_s \geq \mathbf{E}\left[\delta\xi\Gamma_{s,T} + \int_s^T \delta f_r \Gamma_{s,r} dr / \mathcal{F}_s \right] \geq 0.$$

We conclude that $Y_s^1 \geq Y_s^2$ for all $0 \leq s \leq T$–a.s. ■

4. Anticipation Financial Markets and BSDENMs

4.1. **Model for ordinary agent**

Recall that we are working in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions. $T > 0$ is a fixed finite time horizon. We assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_s = \mathcal{F}_T = \mathcal{F}$ for all $s \geq T$. For the ordinary agent with information flow $\mathbf{F}$, the discounted asset prices S are semi-martingale. We assume the financial market to be *free of arbitrage* and *complete* in the following sense:
• (*free of arbitrage*) there exists a probability measure $Q^{\mathbf{F}}$ equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{F}_T)$ such that S is a local $(Q^{\mathbf{F}}, \mathbf{F})$–martingale;
• (*complete*) any bounded $\mathcal{F}_T$–measurable random variable can be written as a sum of a constant and a stochastic integral with respect to S.
Let us define the data (ξ, f) as follows:

Condition 4.1.
1. ξ is a $\mathcal{F}_T$–measurable random variable such that $\mathbf{E}_{\mathbf{Q}^{\mathbf{F}}} e^{\lambda A_T} |\xi|^2 < \infty$.
2. $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that
(a) $(f(t,\omega,0,0))_{t \leq T}$ is $\mathcal{F}$–progressively measurable and $\mathbf{E}_{\mathbf{Q}^{\mathbf{F}}} \int_0^T e^{\lambda A_t} \frac{|f(t,0,0)|^2}{\alpha_t^2} dt < \infty$.
(b) There exists two non-negative $\mathcal{F}_t$–adapted processes $(u_t)_{t \leq T}$ and $(v_t)_{t \leq T}$ such that: for all $t \in [0,T]$ and $y, y', z, z' \in \mathbb{R}$

$$|f(t,y,z) - f(t,y',z')| \leq u_t |y - y'| + v_t |z - z'|.$$

We suppose also that: $\exists \varepsilon > 0, \quad \alpha_t^2 \geq u_t + |v_t|^2 \geq \varepsilon$.

The non insider BSDENM is given by

$$Y_t^{\mathbf{F}} = \xi + \int_t^T f(s, Y_s^{\mathbf{F}}, Z_s^{\mathbf{F}}) ds - \int_t^T Z_s^{\mathbf{F}} dM_s, \quad 0 \leq t \leq T. \tag{5}$$

Theorem 4.1. *Under Condition 4.1, there exists unique solution of the BSDENM in the non insider space.*

4.2. **Model for insider**

To analyze the effects of anticipative information, we consider in addition to the ordinary investor the insider one, which possess from the beginning additional information about the outcome of some $\mathcal{F}_T$ random variable G with value in a Polish space $(\mathbf{X}, \mathcal{X})$ and therefore has the enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t \in [0,T]}$ with $(\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G))_{t \in [0,T]}$ at his disposal up to time T. It is now well known that under the following

Assumption 4.1. The regular conditional distributions of G given $\mathcal{F}_t$, $t \in [0,T]$, are equivalent to the law of G for $\mathbf{P}$–almost all $\omega \in \Omega$, i.e.

$$\mathbf{P}[G \in \cdot/\mathcal{F}_t](\omega) \sim \mathbf{P}[G \in \cdot], \forall\, t \in [0,T] \text{ and } \mathbf{P} - a.a.\ \omega \in \Omega,$$

there exist the so-called martingale preserving probability measure. The σ–algebras $\mathcal{F}_T$ and $\sigma(G)$ become independent under $Q^{\mathbf{G}}$ and $Q^{\mathbf{G}} = Q^{\mathbf{F}}$ on $(\Omega, \mathcal{F}_T)$ (see [2], Theorem 3.1). Any normal martingale $(M_t)_{0 \le t \le T}$ is still a normal martingale under $(Q^{\mathbf{G}}, \mathbf{G})$ since (see [2], Theorem 3.2)

$$\langle M, M \rangle_t^{(\mathbf{Q^G},\mathbf{G})} = \langle M, M \rangle_t^{(\mathbf{Q^G},\mathbf{F})} = \langle M, M \rangle_t^{(\mathbf{Q^F},\mathbf{F})} = \int_0^t \alpha_s^2 ds.$$

Also, the stochastic integrals $\mathbf{F} - \int \phi_s dM_s$ and $\mathbf{G} - \int \phi_s dM_s$ have a common version (see [2], Proposition 3.4), this imply that M verifies the same structural equation (2) under $(\mathbf{Q^G}, \mathbf{G})$. Moreover, the predictable representation property still holds on $\mathcal{L}^2(\Omega, \mathcal{G}_\tau, Q^{\mathbf{G}})$ (see [2], Theorem 4.2).

The BSDENM relative to insider in the enlarged space is described by

$$Y_t^{\mathbf{G}} = \xi + \int_t^T f(s, Y_s^{\mathbf{G}}, Z_s^{\mathbf{G}}) ds - \int_t^T Z_s^{\mathbf{G}} dM_s, \quad 0 \le t \le T. \tag{6}$$

So we have the result that follows.

Theorem 4.2. *Under Condition 4.1, there exists unique solution of the BSDENM in the insider space.*

Proof. Since $\mathcal{F}_t \subseteq \mathcal{G}_t$, the process $(f(t,.,.))_{t \le T}$ is $\mathcal{G}_t$–progressively measurable and Condition 4.1. 2(b) still holds. Moreover, as $Q^{\mathbf{F}} \sim Q^{\mathbf{G}}$ the $Q^{\mathbf{F}}$–null sets are the same as the $Q^{\mathbf{G}}$–null sets. On the other hand, we have (see [2], Theorem 3.2, Proposition 3.3 and 3.4)

$$L^2(\Omega, \mathcal{F}_T, Q^{\mathbf{F}}) = L^2(\Omega, \mathcal{F}_T, Q^{\mathbf{G}}) \subset L^2(\Omega, \mathcal{G}_T, Q^{\mathbf{G}}).$$

Then, $E_{Q^{\mathbf{G}}} e^{\lambda A_T} |\xi|^2 + E_{Q^{\mathbf{G}}} \int_0^T e^{\lambda A_t} \frac{|f(t,0,0)|^2}{\alpha_t^2} dt < +\infty.$

The data (ξ, f) verifies Condition 4.1. Then, by Theorem 3.2 there exists a unique $(Q^{\mathbf{G}}, \mathcal{G})$–solution of the insider BSDENM (6). ■

4.3. **Comparison of solutions**

Theorem 4.3. *Let* $(Y^{\mathbf{F}}, Z^{\mathbf{F}})$ *and* $(Y^{\mathbf{G}}, Z^{\mathbf{G}})$ *be the solutions of the BSDENMs (5) and (6). If* $f(s,y,z)$ *is a concave function with respect to y and z, then*

$$Y_t^{\mathbf{F}} \le E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t].$$

Remark 4.1. The insider possesses the additional information but he hides it in front of the general public. So, in the reality we compare $Y_t^{\mathbf{F}}$ with $E_{Q^{\mathbf{F}}}[(E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t])/\mathcal{F}_t]$. Indeed, $Q^{\mathbf{F}} = Q^{\mathbf{G}}$ on $(\Omega, \mathcal{F}_T)$ (see [2], Theorem 3.1 (b)), then we have

$$E_{Q^{\mathbf{F}}}\Big[\Big(E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t]\Big)/\mathcal{F}_t\Big] = E_{Q^{\mathbf{G}}}\Big[\Big(E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t]\Big)/\mathcal{F}_t\Big] = E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t].$$

To prove Theorem 4.3, we need the following lemmas.

Lemma 4.1. ([1], Lemma 5.3) *Let F be an integrable G_s-measurable variable* $(s \geq 0)$*, then*

$$E_{Q^{\mathrm{G}}}[F/\mathcal{F}_t] = \mathbf{E}_{Q^{\mathrm{G}}}[F/\mathcal{F}_s] \quad \forall s \leq t \leq T.$$

Lemma 4.2. *For each G–adapted square integrable process Z and for each* $t \in [0,T]$*, the following "compatibility" relation holds.*

$$E_{Q^{\mathrm{G}}}\left[\int_0^t Z_s dM_s/\mathcal{F}_t\right] = \int_0^t E_{Q^{\mathrm{G}}}[Z_s/\mathcal{F}_s]dM_s. \tag{7}$$

Proof. Given in the Appendix. ■

Lemma 4.3. *The pair of process* $(E_{Q^{\mathrm{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t], E_{Q^{\mathrm{G}}}[Z_t^{\mathbf{G}}/\mathcal{F}_t],\ 0 \leq t \leq T)$ *satisfies the following BSDENM*

$$dE_{Q^{\mathrm{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t] = E_{Q^{\mathrm{G}}}[f(t, Y_t^{\mathbf{G}}, Z_t^{\mathbf{G}})/\mathcal{F}_t]dt - E_{Q^{\mathrm{G}}}[Z_t^{\mathbf{G}}/\mathcal{F}_t]dM_t. \tag{8}$$

Proof. The pair $(Y_t^{\mathbf{G}}, Z_t^{\mathbf{G}}, 0 \leq t \leq T)$ is the solution of the BSDENM (6). By considering the $\mathcal{F}_t$–conditional expectation, we have by applying Lemma 4.1

$$dE_{Q^{\mathrm{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t] = -E_{Q^{\mathrm{G}}}[f(t, Y_t^{\mathbf{G}}, Z_t^{\mathbf{G}})/\mathcal{F}_t]dt + E_{Q^{\mathrm{G}}}[Z_t^{\mathbf{G}}/\mathcal{F}_t]dM_t.$$

Which completes the proof. ■

5. Appendix: Some Proofs

To begin, let us introduce the notation of the Malliavin derivative. Recall that the multiple stochastic integral $I_n(f_n)$ is defined as

$$I_n(f_n) = n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\ldots,t_n)dM_{t_1}\ldots dM_{t_1}, \quad f_n \in \mathbf{L}^2_\alpha([0,T])^{\circ^n} \ \ n \geq 1,$$

with

$$\mathbf{E}[I_n(f_n), I_m(g_m)] = \mathbf{I}_{(n=m)} n! \langle f_n, g_n\rangle_{L^2_\alpha([0,T])},$$

where $\mathbf{L}^2_\alpha([0,T])$ is the space of $\alpha^2_{t_1}\ldots\alpha^2_{t_n}dt_1\ldots dt_n$–square integrable symmetric functions. We suppose that $(M_t)_{t\leq T}$ has the chaos representation property (for example when ϕ is deterministic (see [16], Proposition 4) and for $\phi_s = \beta M_{s-}$, $\beta \in [2,0)$ (see [16], Proposition 6). This leads that every $F \in \mathbf{L}^2(\Omega)$ has the representation $F = \sum_{n=0}^\infty I_n(f_n)$. Let $D : Dom(D) \to \mathbf{L}^2(\Omega \times [0,T]; d\mathbf{P} \times dt$ denote the closable unbounded gradient operator defined as

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(.\,,t)), \quad d\mathbf{P} \times dt - \text{a.e.},$$

with $F = \sum_{n=0}^\infty I_n(f_n)$. The adjoin δ of D is defined by the duality relation

$$\mathbf{E}[F\delta(u)] = \mathbf{E}\left[\langle D_. F, u\rangle_{\mathbf{L}^2([0,T])}\right], \quad F \in Dom(D),\ u \in Dom(\delta), \tag{9}$$

and it coincide with the stochastic integral with respect to $(M_t)_{t\leq T}$ for every predictable square integrable process (cf. [24]).

5.1. **Proof of Lemma 4.2**

Let F be a $\mathcal{F}_t$–measurable bounded random variable such that $DF \in \mathbf{L}^2_\alpha(\Omega \times [0,T])$. First, note that, for all $0 \leq t \leq T$, $\mathbf{E}_{\mathbf{Q}^{\mathbf{G}}}\left[F\int_0^t \|Z_s^{\mathbf{G}}\|^2 ds\right] < \infty$. Using the duality relation (9), we obtain

$$E_{Q^{\mathbf{G}}}\left[F\int_0^t Z_s^{\mathbf{G}} dM_s\right] = E_{Q^{\mathbf{G}}}\left[\int_0^t (D_sF)Z_s^{\mathbf{G}} ds\right] = \int_0^t E_{Q^{\mathbf{G}}}[(D_sF)Z_s^{\mathbf{G}}]ds$$

$$= \int_0^t E_{Q^{\mathbf{G}}}[E_{Q^{\mathbf{G}}}[(D_sF)Z_s^{\mathbf{G}}/\mathcal{F}_s]]ds. \tag{10}$$

In the other hand, we can prove, by the same way in proposition 1.2.4 [25], that

$$D_s(E_{Q^{\mathbf{G}}}[F/\mathcal{F}_s]) = E_{Q^{\mathbf{G}}}[D_sF/\mathcal{F}_s], \quad \forall\, 0 \leq s < t \leq T.$$

From Lemma 4.1, we have $E_{Q^{\mathbf{G}}}[F/\mathcal{F}_s] = E_{Q^{\mathbf{G}}}[F/\mathcal{F}_t]$. So, F is $\mathcal{F}_t$–measurable, then $E_{Q^{\mathbf{G}}}[F/\mathcal{F}_t] = F$ and $D_sF = E_{Q^{\mathbf{G}}}[(D_sF)/\mathcal{F}_s]$ which implies that D_sF is $\mathcal{F}_s$-measurable. Combining this and (10), we obtain

$$E_{Q^{\mathbf{G}}}\left[F\int_0^t Z_s^{\mathbf{G}} dM_s\right] = \int_0^t E_{Q^{\mathbf{G}}}\left[(D_sF)E_{Q^{\mathbf{G}}}[Z_s^{\mathbf{G}}/\mathcal{F}_s]\right]ds$$

$$= E_{Q^{\mathbf{G}}}\left[\int_0^t (D_sF)E_{Q^{\mathbf{G}}}[Z_s^{\mathbf{G}}/\mathcal{F}_s]\right]ds = E_{Q^{\mathbf{G}}}\left[F\int_0^t {}_sE_{Q^{\mathbf{G}}}[Z_s^{\mathbf{G}}/\mathcal{F}_s]dM_s\right].$$

The result of the lemma thus follows. ■

5.1. **Proof of Theorem 4.3**

The function $f(t,\cdot,\cdot)$ is a concave, then

$$f\left(t, E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t], E_{Q^{\mathbf{G}}}[Z_t^{\mathbf{G}}/\mathcal{F}_t]\right) \leq E_{Q^{\mathbf{G}}}[f(t, Y_t^{\mathbf{G}}, Z_t^{\mathbf{G}})/\mathcal{F}_t] \quad Q^{\mathbf{G}} - a.s.\ 0 \leq t \leq T,$$

and $E_{Q^{\mathbf{G}}}[f(s, Y_s^{\mathbf{G}}, Z_s^{\mathbf{G}})/\mathcal{F}_s]$ is the drift of (8). Combining this and (7), we can write (8) as

$$-dE_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t] = E_{Q^{\mathbf{G}}}[f(t, Y_t^{\mathbf{G}}, Z_t^{\mathbf{G}})/\mathcal{F}_t]dt - E_{Q^{\mathbf{G}}}[Z_t^{\mathbf{G}}/\mathcal{F}_t]dM_t.$$

We complete the proof by applying the Comparison Theorem to $(Y_t^{\mathbf{F}}, Z_t^{\mathbf{F}})$ and$(E_{Q^{\mathbf{G}}}[Y_t^{\mathbf{G}}/\mathcal{F}_t], E_{Q^{\mathbf{G}}}[Z_t^{\mathbf{G}}/\mathcal{F}_t])$. ■

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