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Almost Sure Exponential Stability for Some Stochastic Neutral Partial Integro-Differential Equations

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Abstract. *This paper is concerned with the dynamics of a delay stochastic neutral integro-differential equation in Hilbert spaces by using the theory of resolvent operator. After establishing a result ensuring the existence and uniqueness of a mild solution of this class of equations, we investigate the exponential stability of the moments of a mild solution as well as its sample paths. An example is given to illustrate the results.*

Key words: Resolvent Operators, C_0-semigroup, Neutral Stochastic Functional Integro-Differential Equations, Wiener Process, Finite Delay, Mild Solutions, Exponential Stability.

AMS Subject Classifications: 34K40, 35B35, 60H15, 93E15

1. Introduction

We investigate the stability of a stochastic neutral partial functional integro-differential equation in a real separable Hilbert space X of the following form:

$$\begin{cases} d[x(t) + h(t,\pi_t x)] = A[x(t) + h(t,\pi_t x)] + \int_0^t B(t-s)[x(s) + h(s,\pi_s x)]ds + f(t,\pi_t x))dt \\ \qquad\qquad + g(t,\pi_t x)dw(t), \quad t > 0 \\ x(t) = \varphi(t), \quad t \in [-r,0], \end{cases} \tag{1}$$

where $\pi_t x : [0,r] \to \mathrm{X}$ is defined by $\pi_t x(s) = x(t-r+s) : 0 \leq s \leq r$. Let $A : D(A) \subseteq \mathrm{X} \to \mathrm{X}$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ defined on X. Let the function $f(t,u), h(t,u)$ and $g(t,u)$ be defined as follows : $f : \mathbb{R}^+ \times \mathcal{C}_t \to \mathrm{X}$, where $\mathbb{R}^+ = [0,\infty), h : \mathbb{R}^+ \times \mathcal{C}_t \to D(A)$ and $g : \mathbb{R}^+ \times \mathcal{C}_t \to \mathcal{L}(\mathrm{Y},\mathrm{X})$ are Borel measurable ; and for each (t,u) are measurable with respect to the σ-algebra $\mathcal{F}_{0,t}(dw)$. In the sequel, we introduce the concept of a mild solution of equation (1) following the deterministic case, Hernandez and Henriquez [15].

Stochastic delay differential equations (SDDEs) play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. In the past few decades, qualitative theory of SDDEs have been studied intensively by many scholars. Here, we refer to [8] and references therein. In recent years, existence, uniqueness, stability, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by several authors. Many important results have been reported, for instance, in [1, 3, 12, 19]. In particular, Caraballo extended in [2] the results from Haussmann [16]; Mao proved in [20] the mean-square exponential stability for the strong solutions of linear stochastic delay equations with finite constant delay, by using the method developed in [16, 17]. Following the ideas of Haussmann [16] and Ichikawa [17], Caraballo and Real [1] considered the stability of the strong solutions of semilinear stochastic delay evolution equations. Recently, Zhang and Ruan [24] obtained the exponential stability of mild solutions to a class of neutral stochastic functional partial differential equations driven by Brownian motion and fractional Brownian motion.

Motivated by the facts stated in the above discussion, in this work we aim to study the almost sure exponential stability for stochastic neutral partial functional integro-differential equation on Hilbert space. We prove that the mild solution to a class of stochastic partial integro-differential equations with delays exists, is unique and also pth moment exponentially stable, by using a fixed point argument.

The stochastic partial differential and integro-diffenrential equations have attracted much attention because of their practical applications in many areas such as physics, chemistry, economics, social sciences, finance, population dynamics, electrical engineering, medicine biology, ecology and other areas of science and engineering. Qualitative properties such as existence, uniqueness, optimality conditions, controllability and stability for various linear and nonlinear stochastic partial differential and integro-differential equations have been extensively studied by many researchers, see for instance [4, 5, 6, 9] and the references therein. Due to the presence of the integral term in our equation, we need to use the theory of resolvent operators as developed by Grimmer [11] instead of using strongly continuous semigroups.

The advantage of using this method is that one can prove at the same time not only the existence and uniqueness of solution of the problem, but also the exponential stability in the pth moment. It is worth noticing that for this reason, the set of assumptions that we have to impose may seem more restrictive than the ones which might be sufficient to ensure the existence and uniqueness of solution, but as we are interested in the stability of solutions we prefer to state all the assumptions needed for that at one stage.

The rest of this work is organized as follows. In Section 2, we introduce some notations and preliminaries, which are used throughout this paper, as well as the definition of the mild solution for stochastic evolution equation problem (1) is also given. In Section 3, we state and prove the existence and uniqueness of mild solutions for stochastic evolution equation problem (1). In Section 4, we discuss the exponential stability in the pth moment of a mild solution of equation (1). Almost sure exponential stability of sample path are considered in Section 4. Finally we illustrate our result by analyzing an example.

2. Preliminaries

Let Y be real separable Hilbert space and $\mathcal{L}(\mathrm{Y},\mathrm{X})$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use the same notation $\|.\|$ to denote the norms in X,Y and $\mathcal{L}(\mathrm{X},\mathrm{Y})$ and use $(.,.)$ to denote inner-product of X and Y without any confusion. For the question of existence of mild solution of the integro-differential equation (1), we recall some fundamental results needed. Regarding the theory of resolvent operators, we refer the reader to [11]. Let Z be the Banach space $\mathcal{D}(A)$ equipped with the graph norm given by $\|y\|_Z := \|Ay\| + \|y\|$ for $y \in Z$. The notation $\mathcal{C}(\mathbb{R}^+; Z)$ stands for the space of all continuous functions from $\mathbb{R}^+$ into Z. We consider the following Cauchy problem :

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds & \text{for } t \geq 0 \\ v(0) = v_0 \in X. \end{cases} \tag{2}$$

Definition [11] **2.1**. A resolvent operator for equation (1) is a bounded linear operator-valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$, having the following properties :
1. $R(0) = I$ (the identity map of X) and $\|R(t)\| \leq Ne^{\eta t}$ for some constants $N > 0$ and $\eta \in \mathbb{R}$.
2. For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.
3. $R(t) \in \mathcal{L}(Z)$ for $t \geq 0$. For $x \in Z, R(.)x \in \mathcal{C}^1(\mathbb{R}^+; X) \bigcap \mathcal{C}(\mathbb{R}^+; Z)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad \textit{for } t \geq 0. \end{aligned}$$

In this work, we shall throughout assume that:
(R1) The operator A is the infinitesimal generator of a C_0-semigroup $(S(t))_{t \geq 0}$ on X.
(R2) For all $t \geq 0$, $B(t)$ is closed linear operator from $\mathcal{D}(A)$ to X and $B(t) \in \mathcal{L}(Z, X)$. For any $z \in Z$, the map $t \mapsto B(t)z$ is bounded, differentiable and the derivative $t \mapsto B'(t)z$ is bounded uniformly continuous on $\mathbb{R}^+$.

The next theorem gives a satisfactory answer to the problem of existence of solutions.

Theorem [11] **2.1**. *Assume that* $(R1) - (R2)$ *hold. Then there exists a unique resolvent operator for the Cauchy problem (2).*

Then in the following, we give some results for the existence of solutions to the integro-differential equation:

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t) & \text{for } t \geq 0 \\ v(0) = v_0 \in X, \end{cases} \tag{3}$$

where $q : \mathbb{R}^+ \to X$ is a continuous function.

Definition [11] **2.2**. A continuous function $v : \mathbb{R}^+ \to X$ is said to be a strict solution of equation (3) if $v \in \mathcal{C}^1(\mathbb{R}^+; X) \bigcap \mathcal{C}(\mathbb{R}^+; Z)$ and v satisfies equation (3).

Theorem [11] **2.2**. *Assume that* $(R1) - (R2)$ *hold. If v is a strict solution of equation (3), then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \text{for } t \geq 0.$$

The resolvent operator plays an important role in the study of the existence of solutions and gives a variation of constants formula for semilinear systems. For more details on resolvent operators, we refer to [11, 13]. Let $\sigma_2(Y, X)$ denote the space of all Hilbert-Schmidt operators from Y to X with the Hilbert-Schmidt norm $\|.\|_2$. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete probability space with an increasing right continuous family $\{\mathcal{F}_t\}_{t\geq 0}$ of complete sub-σ-algebras of $\mathcal{F}$. Let $\beta_n(t)(n=1,2,3,\ldots)$ be a sequence of real-valued standard Brownian motions mutually independent defined on this probability space. Set

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \beta_n(t) e_n, \quad t \geq 0,$$

where $\lambda_n \geq 0(n=1,2,3,\ldots)$ are nonnegative real numbers and $\{e_n\}(n=1,2,3,\ldots)$ is a complete orthonormal basis in Y . Let $Q \in \mathcal{L}(\mathrm{Y},\mathrm{Y})$ be an operator defined by $Qe_n = \lambda_n e_n$. The above Y-valued stochastic process $w(t)$ is called a Q-Wiener process. Now, we define a real-valued stochastic integral of Y-valued $\mathcal{F}_t$ –adapted predictable process $\ell(t)$ with respect to the Q-Wiener process $w(t)$.

Definition 2.3. Let $\ell(t)$ be a Y-valued $\mathcal{F}_t$-adapted predictable process such that $\mathbf{E}\int_0^t \|\ell(s)\|^2 ds < \infty$ for any $t \in [0,\infty)$. Then, we define the real-valued stochastic integral $\int_0^t \langle \ell(s), dw(s)\rangle$ by

$$\int_0^t \langle \ell(s), dw(s)\rangle = \sum_{n=1}^{\infty} \int_0^t (\ell(s), e_n) dw(s) e_n,$$

where $w(s)e_n = (w(s), e_n) = \sqrt{\lambda_n}\, \beta_n(s)$.

Definition 2.4. Let $\ell(t)$ be a $\mathcal{L}(\mathrm{X},\mathrm{Y})$-valued function and let λ be a sequence $\{\sqrt{\lambda_1}, \sqrt{\lambda}_2, \ldots\}$ Then we define

$$\|\ell(t)\|_\lambda = \left\{ \sum_{n=1}^{\infty} \| \sqrt{\lambda_n}\, \ell(t) e_n \|^2 \right\}^{1/2}.$$

If $\|\ell(t)\|_\lambda^2 < \infty$, then $\ell(t)$ is called λ –Hilbert-Schmidt operator and let $\sigma(\lambda)(\mathrm{Y},\mathrm{X})$ denote the space of all λ –Hilbert-Schmidt operators from Y to X.

Lemma [22] **2.1**. *If trace $Q < \infty$ and $\ell(t) \in L(X,Y)$, then*
$\|\ell(t)\|_\lambda^2 \leq (trace\ Q)\|\ell(t)\|^2$.
If $0 \leq \lambda_j \leq \mu$ for all $j \geq 1$ and $\ell(t) \in \sigma_2(X,Y)$, then $\|\ell(t)\|_\lambda^2 \leq \mu \|\ell(t)\|_2^2$.

Next, we define the X-valued stochastic integral with respect to the Y-valued Q- Wiener process $w(t)$, see [10, 23] for more details.

Definition 2.5. Let $\Phi : [0,\infty) \to \sigma(\lambda)(\mathrm{Y},\mathrm{X})$ be a predictable, $\mathcal{F}_t$ –adapted process. Then, for any Φ satisfying $\int_0^t \mathbf{E}\|\Phi(s)\|_\lambda^2 ds < \infty$ we define the X –valued stochastic integral $\int_0^t \Phi(s) dw(s) \in \mathrm{X}$ with respect to $w(t)$ by

$$\left(\int_0^t \Phi(s) dw(s), y\right) = \int_0^t \langle \Phi^*(s) y, dw(s)\rangle, \quad y \in \mathrm{X},$$

where Φ^* is the adjoint operator of Φ.

Given a stochastic process $\{x(t), t \geq 0\}, \mathcal{F}_{r,s}(x)$ denotes the least sub–σ –algebra of the σ –algebra $\mathcal{F}$ generated by $\{x(t), r \leq t \leq s\}$. $\mathcal{F}_{r,s}(dw)$ denotes the least sub–σ –algebra of $\mathcal{F}$ generated by the increments $\{w(s) - w(t), r \leq t \leq s\}$ of the Y –valued Q –Wiener process

$\{w(t), t \geq 0\}$ with $w(0) = 0$. The least sub–σ–algebra of F corresponding to the past process $\{\varphi(t), t \in [-r, 0]\}(r > 0)$ is denoted by $\mathcal{F}_{-r,0}(\varphi)$ with respect to which φ is measurable and not be dependent on $\mathcal{F}_{0,\infty}(dw)$. The least sub–$\sigma$–algebra that contains the σ–algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ is denoted by$\mathcal{F}_1 \vee \mathcal{F}_2 \vee \ldots \vee \mathcal{F}_n$.
Let $\mathcal{C}_t$ be the space of continuous functions $x : [-r, t] \to \mathrm{X}(0 \leq t < \infty)$ with the norm $\|x\|_t = \sup_{-r \leq s \leq t} \|x(s)\|$. The past process φ has almost sure (a.s.) continuous paths and $\mathbf{E}\|\varphi(., w)\|_0^p < \infty, p \geq 2$. Let $\mathfrak{B}_T = \mathfrak{B}_T(\varphi)$ be the space of measurable random processes $\phi(t, w)$ with a.s. continuous paths ; for each $t \in [0, T], \phi(t, w)$ is measurable with respect to $\mathcal{F}_{0,t}(dw)$ and $\phi(s, w) = \varphi(s, w)$ for $-r \leq s \leq 0$ with the norm $\|\phi\|_{\mathfrak{B}_t} = (\mathbf{E}\|\phi(., w)\|_T^p)^{1/p}$.

The following lemma is due to Rodkina [21].

Lemma [21] **2.2**. *$\mathfrak{B}_T$ is a Banach space.*

Definition 2.6. A resolvent operator $\{R(t), t \geq 0\}$ is said to be exponentially stable if there exist positive constants M, a such that

$\|R(t)\| \leq M \exp(-at), \quad t \geq 0,$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(\mathrm{X}) = \mathcal{L}(\mathrm{X}, \mathrm{X})$.

Definition 2.7. A stochastic process $\{x(t), t \in [-r, T]\}(0 < T < \infty)$ is called a mild solution of equation (1) if

(i) $x(t)$ is $\mathcal{F}_t$-adapted and predictable with $\int_0^T \|x(t)\|^2 dt < \infty$, a.s.,

(ii) $x(t) = \varphi(t), t \in [-r, 0]$ a.s. and $\mathcal{F}_{-r,t}(x) \vee \mathcal{F}_{0,t}(dw)$ does not depend on $\mathcal{F}_{t,\infty}(dw)$ for each $t \in [0, T]$, and

(iii) $x(t)$ satisfies the following integral equation

$$x(t) = R(t)(\varphi(0) + h(0, \varphi)) - h(t, \pi_t x) + \int_0^t R(t-s) f(s, \pi_s x) ds$$

$$+ \int_0^t R(t-s) g(s, \pi_s x) dw(s), \quad a.s., \quad t \in [0, T].$$

Moreover, we shall state the following lemmas that will be needed in the sequel.

Lemma [7] **2.3**. *Let Φ be a predictable process. For any arbitrary $p > 2$ there exists a constant $c = c(p, T) > 0$ such that for any $T > 0$ and a proper modification of the stochastic convolution $W_A^\Phi(t) := \int_0^t R(t-s)\Phi(s) dw(s)$ one has:*

$$\mathbf{E}\left\{\sup_{t \leq T} \left\| \int_0^t R(t-s)\Phi(s) dw(s) \right\|^p \right\} \leq c(p, T) \sup_{t \leq T} \|R(t)\|^p \mathbf{E} \int_0^T \|\Phi(s)\|_\lambda^p ds.$$

Lemma [7] **2.4**. *Let $\Phi : [0, +\infty) \to \sigma(\lambda)(\mathrm{Y}, \mathrm{X})$ be a predictable, F_t-adapted process with $\int_0^t E\|\Phi(s)\|_\lambda^p ds < \infty$ for some integer $p > 2$ and any $t \geq 0$. Then there exists a constant k_p such that for any natural number N*

$$\mathbf{E}\left\{\sup_{N\le t\le N+1}\left\|\int_N^t R(t-s)\Phi(s)dw(s)\right\|_{\mathrm{X}}^p\right\}\le k_pM^p\int_N^{N+1}\mathbf{E}\|\Phi(s)\|_\lambda^p ds.$$

3. Existence of a Unique Solution

In this section, we discuss the existence and uniqueness of a mild solution of equation (1). Let the following assumptions hold a.s. :

(**H1**) For $p > 2$, the functions $f(t,u)$ and $g(t,u)$ satisfy the Lipschitz and linear growth conditions:

$$\|f(t,\pi_t x)-f(t,\pi_t y)\|^p\le C_1\|x-y\|_t^p,\quad C_1>0;$$
$$\|g(t,\pi_t x)-g(t,\pi_t y)\|_\lambda^p\le C_2\|x-y\|_t^p,\quad C_2>0;$$
$$\|f(t,\pi_t x)\|^p+\|g(t,\pi_t x)\|_\lambda^p\le C_3(1+\|x\|_t^p),\quad C_3>0;$$

where $x,y:[-r,t]\to \mathrm{X}$ are continuous and $t\ge 0,\pi_t x,\pi_t y$ belong to $\mathcal{C}_t$.

(**H2**) $h(t,u)$ is a continuous function in t and satisfies:

$$\|h(t,\pi_t x)-h(t,\pi_t y)\|\le C_4\|x-y\|_t,\quad C_4>0;$$
$$\|h(t,u)\|\le C_5$$

where $C_5=\sup_{(t,u)\in\mathbb{R}^+\times\mathcal{C}_t}\|h(t,u)\|$ and $\pi_t x,\pi_t y\in\mathcal{C}_t$.

Noting that $M_T=\sup_{t\in[0,T]}\|R(t)\|_{\mathcal{L}(\mathrm{X})}<\infty$, then the following iterative procedure will be used in the proof of our result: Define for each integer $n=1,2,3,\ldots,$

$$\begin{aligned}x_n(t)&=R(t)(\varphi(0)+h(0,\varphi))-h(t,\pi_t x_n)+\int_0^t R(t-s)f(s,\pi_s x_{n-1})ds\\&\quad+\int_0^t R(t-s)g(s,\pi_s x_{n-1})dw(s),a.s.,\quad t\in[0,T],\end{aligned}\tag{4}$$

and for $n=0$,

$$x_0(t)=R(t)\varphi(0),\quad t\in[0,T],\tag{5}$$

while for $n=0,1,2,3,\ldots,$

$$x_n(t)=\varphi(t),\quad t\in[-r,0].$$

It should be noted that the above scheme is not explicit as the function h on right-hand side of equation (4) depends explicitly on x_n. But, this seems to be standard when one considers neutral equations. In other words, such a procedure is well-defined, that is, with an initial approximation $x_0\in\mathfrak{B}_T$ satisfying equation (5), generate x_1 implicitly using the iterative procedure (4) and so on. Clearly, this procedure boils down to the one in [14, 18, 22] for the special case $h=0$.

Theorem 3.1. *Let the assumptions* $(H1)$ - $(H2)$ *hold. Then there exists a unique mild solution* $x(t)$ *of equation (1) in* B_T *provided that* $C_4<1$.

Proof. Let T be any fixed time with $0<T<\infty$. To begin with, suppose that assumptions (**H1**) and (**H2**) hold. We have

$$x_n(s) = R(s)(\varphi(0) + h(0,\varphi)) - h(s,\pi_s x_n) + \int_0^s R(s-\tau) f(\tau, \pi_\tau x_{n-1}) d\tau$$

$$+ \int_0^s R(s-\tau) g(\tau, \pi_\tau x_{n-1}) dw(\tau).$$

Exploiting assumptions (**H1**) and (**H2**), we obtain the following estimations :

$$\begin{aligned}
\|x_n(s)\| \le\ & \|R(s)\| \|(\varphi(0) + h(0,\varphi))\| + \|h(s,\pi_s x_n)\| + \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| \\
& + \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\| \\
\le\ & \|R(s)\| \|\varphi(0)\| + \|R(s)\| \|h(0,\varphi) - h(0,0) + h(0,0)\| + \|h(s,\pi_s x_n) - h(s,0) + h(s,0)\| \\
& + \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| + \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\| \\
\le\ & \|R(s)\| \|\varphi\|_0 + \|R(s)\| \|h(0,\varphi) - h(0,0)\| + \|R(s)\| \|h(0,0)\| \\
& + \|h(s,\pi_s x_n) - h(s,0)\| + \|h(s,0)\| + \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| \\
& + \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\| \\
\le\ & \|R(s)\| \|\varphi\|_0 + \|R(s)\| C_4 \|\varphi\|_0 + \|R(s)\| C_4 + C_4 \|x_n\|_s + C_5 \\
& + \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| \\
& + \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\| \\
\le\ & \|R(s)\| \|\varphi\|_0 (1 + C_4) + C_5(\|R(s)\| + 1) + C_4 \|x_n\|_s + \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| \\
& + \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\|.
\end{aligned}$$

Which gives

$$\begin{aligned}
\sup_{0\le s\le t} \|x_n(s)\| \le\ & \sup_{0\le s\le t} \|R(s)\| \|\varphi\|_0 (1 + C_4) + C_5 (\sup_{0\le s\le t} \|R(s)\| + 1) + C_4 \sup_{0\le s\le t} \|x_n\|_s \\
& + \sup_{0\le s\le t} \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| + \sup_{0\le s\le t} \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\| \\
\le\ & M_T \|\varphi\|_0 (1 + C_4) + C_5 (M_T + 1) + C_4 \sup_{0\le s\le t} \|x_n\|_t \\
& + \sup_{0\le s\le t} \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\| + \sup_{0\le s\le t} \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\|
\end{aligned}$$

Therefore, we get that

$$(1 - C_4) \|x_n\|_t \le M_T \|\varphi\|_0 (1 + C_4) + C_5 (M_T + 1) + \sup_{0\le s\le t} \left\| \int_0^s R(s-\tau) f(\tau,\pi_\tau x_{n-1}) d\tau \right\|$$

$$+ \sup_{0\le s\le t} \left\| \int_0^s R(s-\tau) g(\tau,\pi_\tau x_{n-1}) dw(\tau) \right\|.$$

It follows that

$$(1-C_4)^p\|x_n\|_t^p \le 4^{p-1}\Big\{M_T^p\|\varphi\|_0^p(1+C_4)^p + C_5^p(M_T+1)^p + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_{n-1})d\tau\Big\|^p$$
$$+\sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_{n-1})dw(\tau)\Big\|^p\Big\}.$$

Taking the expectation, we get the following inequality

$$(1-C_4)^p\mathbf{E}\|x_n\|_t^p \le 4^{p-1}\{M_T^p\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + C_5^p(M_T+1)^p$$
$$+\mathbf{E}\Big\{\sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_{n-1})d\tau\Big\|^p\Big\}$$
$$+\mathbf{E}\Big\{\sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_{n-1})dw(\tau)\Big\|^p\Big\}\Big\}.$$

Applying Lemma 2.3 yields

$$(1-C_4)^p\mathbf{E}\|x_n\|_t^p \le 4^{p-1}\{M_T^p\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + C_5^p(M_T+1)^p$$
$$+\ T^{p-1}M_T^p\mathbf{E}\sup_{0\le s\le t}\int_0^s\|f(\tau,\pi_\tau x_{n-1})\|^p d\tau$$
$$+c(p,T)\sup_{0\le s\le t}\|R(s)\|^p\mathbf{E}\int_0^t\|g(\tau,\pi_\tau x_{n-1})\|_\lambda^p d\tau\Big\}$$
$$\le 4^{p-1}\Big\{M_T^p\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + C_5^p(M_T+1)^p + M_T^pT^{p-1}\mathbf{E}\int_0^t\|f(s,\pi_s x_{n-1})\|^p ds$$
$$+c(p,T)M_T^p\mathbf{E}\int_0^t\|g(s,\pi_s x_{n-1})\|_\lambda^p ds\Big\}$$
$$\le 4^{p-1}\{M_T^p\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + C_5^p(M_T+1)^p$$
$$+M_T^p(T^{p-1}+c(p,T))\int_0^t\mathbf{E}(\|f(s,\pi_s x_{n-1})\|^p + \|g(s,\pi_s x_{n-1})\|_\lambda^p)ds\}.$$

By assumption (**H1**), we obtain that

$$\mathbf{E}\|x_n\|_t^p \le \frac{4^{p-1}}{(1-C_4)^p}\{M_T^p\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + C_5^p(M_T+1)^p\}$$
$$+\frac{4^{p-1}M_T^p(T^{p-1}+c(p,T))C_3}{(1-C_4)^p}\times\int_0^T(1+\mathbf{E}\|x_{n-1}\|_s^p)ds$$

for $n = 1, 2, 3, \cdots$
Recalling that

$$\mathbf{E}\|x_0\|_t^p = \mathbf{E}\Big(\sup_{-r\le s\le t}\|x_0(s)\|\Big)^p,$$

we may write

$$\mathbf{E}\left(\sup_{0\le s\le t}\|x_0(s)\|\right)^p \le \mathbf{E}\left(\sup_{0\le s\le t}\|R(s)\|\,\|\varphi(0)\|\right)^p \le M_T^p\mathbf{E}\|\varphi(0)\|^p \le M_T^p\mathbf{E}\|\varphi\|_0^p < \infty,$$
$$\mathbf{E}\left(\sup_{-r\le s\le 0}\|x_0(s)\|\right)^p \le \mathbf{E}\left(\sup_{-r\le s\le 0}\|\varphi(s)\|\right)^p \le \mathbf{E}\|\varphi\|_0^p < \infty.$$

So, we obtain

$$\mathbf{E}\|x_0\|_t^p \le \max(M_T,1)\mathbf{E}\|\varphi\|_0^p < \infty.$$

Then, we obtain that

$$\mathbf{E}\|x_n\|_t^p < \infty,$$

for all $n = 1,2,3,\cdots$, $t \in [0,T]$. Thus the sequence $\{x_n\}$ belongs to $\mathfrak{B}_T$.

Now, we show that $\{x_n\}$ is Cauchy in $\mathfrak{B}_T$. For this, consider for $s \in [0,T]$,

$$\begin{aligned} x_1(s)-x_0(s) &= R(s)(\varphi(0)+h(0,\varphi)) - h(s,\pi_s x_1) + \int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)ds \\ &\quad + \int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau) - R(s)\varphi(0) \\ &= R(s)h(0,\varphi) - (h(s,\pi_s x_1)-h(s,\pi_s x_0)) - h(s,\pi_s x_0) \\ &\quad + \int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau + \int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau). \end{aligned}$$

By assumptions (**H1**) and (**H2**), we have

$$\begin{aligned} \|x_1(s)-x_0(s)\| \le\ & \|R(s)h(0,\varphi)\| + \|h(s,\pi_s x_1)-h(s,\pi_s x_0)\| + \|h(s,\pi_s x_0)\| \\ &+ \left\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\right\| + \left\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\right\| \\ \le\ & \|R(s)\|\,\|h(0,\varphi)-h(0,0)+h(0,0)\| + \|h(s,\pi_s x_1)-h(s,\pi_s x_0)\| \\ &+ \|h(s,\pi_s x_0)-h(s,0)+h(s,0)\| + \left\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\right\| \\ &+ \left\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\right\| \\ \le\ & \|R(s)\|\,\|h(0,\varphi)-h(0,0)\| + \|R(s)\|\,\|h(0,0)\| + \|h(s,\pi_s x_1)-h(s,\pi_s x_0) \\ &+ \|h(s,\pi_s x_0)-h(s,0)\| + \|h(s,0)\| + \left\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\right\| \\ &+ \left\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\right\| \\ \le\ & \|R(s)\|C_4\|\varphi\|_0 + \|R(s)\|C_5 + C_4\|x_1-x_0\|_s + C_4\|x_0\|_s + C_5 \\ &+ \left\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\right\| + \left\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\right\|, \end{aligned}$$

which gives

$$\begin{aligned}
\sup_{0\le s\le t}\|x_1(s)-x_0(s)\| &\le \sup_{0\le s\le t} C_4\|R(s)\|\,\|\varphi\|_0 + C_5\sup_{0\le s\le t}\|R(s)\| + \sup_{0\le s\le t} C_4\|x_1-x_0\|_s \\
&\quad + C_4\sup_{0\le s\le t}\|x_0\|_s + C_5 + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\Big\| \\
&\quad + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\Big\| \\
&\le M_T C_4\|\varphi\|_0 + C_5(M_T+1) + \sup_{0\le s\le t} C_4\|x_0\|_t + \sup_{0\le s\le t} C_4\|x_1-x_0\|_t \\
&\quad + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\Big\| + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\Big\|.
\end{aligned}$$

As before, one has the following inequality

$$\begin{aligned}
(1-C_4)\|x_1-x_0\|_t &\le C_4(M_T\|\varphi\|_0 + \|x_0\|_t) + C_5(M_T+1) \\
&\quad + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\Big\| + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\Big\|
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(1-C_4)^p\|x_1-x_0\|_t^p &\le 4^{p-1}\{C_4^p(M_T\|\varphi\|_0 + \|x_0\|_t)^p + C_5^p(M_T+1)^p \\
&\quad + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\Big\|^p \\
&\quad + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\Big\|^p\Big\}.
\end{aligned}$$

Taking the expectation, in both sides of the above inequality, yields

$$\begin{aligned}
(1-C_4)^p\mathbf{E}\|x_1-x_0\|_t^p &\le 4^{p-1}\{2^{p-1}C_4^p(M_T^p\mathbf{E}\|\varphi\|_0^p + \mathbf{E}\|x_0\|_t^p) + C_5^p(M_T+1)^p\} \\
&\quad + 4^{p-1}\mathbf{E}\Big\{\sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x_0)d\tau\Big\|^p\Big\} \\
&\quad + 4^{p-1}\mathbf{E}\Big\{\sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x_0)dw(\tau)\Big\|^p\Big\}.
\end{aligned}$$

Apply once again Lemma 2.3 to get

$$\begin{aligned}
(1-C_4)^p\mathbf{E}\|x_1-x_0\|_t^p &\le 4^{p-1}\{2^{p-1}C_4^p(M_T^p\mathbf{E}\|\varphi\|_0^p + \mathbf{E}\|x_0\|_t^p) + C_5^p(M_T+1)^p \\
&\quad + \mathbf{E}T^{p-1}M_T^p\sup_{0\le s\le t}\int_0^s\|f(\tau,\pi_\tau x_0)\|^p d\tau \\
&\quad + c(p,T)\sup_{0\le s\le t}\|R(s)\|^p\mathbf{E}\int_0^t\|g(\tau,\pi_\tau x_0)\|_\lambda^p d\tau\Big\} \\
&\le 4^{p-1}\{2^{p-1}C_4^p(M_T^p\mathbf{E}\|\varphi\|_0^p + \mathbf{E}\|x_0\|_t^p) + C_5^p(M_T+1)^p \\
&\quad + M_T^pT^{p-1}\mathbf{E}\int_0^t\|f(s,\pi_s x_0)\|^p ds \\
&\quad + c(p,T)M_T^p\mathbf{E}\int_0^t\|g(s,\pi_s x_0)\|_\lambda^p ds\Big\}
\end{aligned}$$

$$\leq 4^{p-1}\{2^{p-1}C_4^p(M_T^p\mathbf{E}\|\varphi\|_0^p + \mathbf{E}\|x_0\|_t^p) + C_5^p(M_T+1)^p + M_T^p(T^{p-1}+c(p,T))\int_0^t \mathbf{E}(\|f(s,\pi_s x_0)\|^p + \|g(s,\pi_s x_0)\|_\lambda^p)ds\Big\}.$$

Next, by assumption (**H1**), we have

$$\mathbf{E}\|x_1 - x_0\|_t^p \leq \frac{4^{p-1}}{(1-C_4)^p} \times \{2^{p-1}C_4^p(M_T^p\mathbf{E}\|\varphi\|_0^p + \mathbf{E}\|x_0\|_t^p) + C_5^p(M_T+1)^p\} + \frac{4^{p-1}M_T^p(T^{p-1}+c(p,T))C_3}{(1-C_4)^p} \times \int_0^T (1+\mathbf{E}\|x_0\|_s^p)ds. \tag{6}$$

Then consider for $s \in [0,T]$

$$x_n(s) - x_{n-1}(s) = h(s,\pi_s x_{n-1}) - h(s,\pi_s x_n) + \int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau + \int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau),$$

to arrive at the following inequality

$$\begin{aligned}\|x_n(s) - x_{n-1}(s)\| \leq\ & \|h(s,\pi_s x_{n-1}) - h(s,\pi_s x_n)\| + \left\|\int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau\right\| \\ & + \left\|\int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau)\right\| \\ \leq\ & C_4\|x_n - x_{n-1}\|_s + \left\|\int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau\right\| \\ & + \left\|\int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau)\right\|.\end{aligned}$$

That implies

$$\begin{aligned}\sup_{-r\leq s\leq t}\|x_n(s) - x_{n-1}(s)\| \leq\ & \sup_{0\leq s\leq t} C_4\|x_n - x_{n-1}\|_s \\ & + \sup_{0\leq s\leq t}\left\|\int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau\right\| \\ & + \sup_{0\leq s\leq t}\left\|\int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau)\right\| \\ \leq\ & \sup_{0\leq s\leq t} C_4\|x_n - x_{n-1}\|_t \\ & + \sup_{0\leq s\leq t}\left\|\int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau\right\| \\ & + \sup_{0\leq s\leq t}\left\|\int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau)\right\| \\ \leq\ & C_4\|x_n - x_{n-1}\|_t + \sup_{0\leq s\leq t}\left\|\int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau\right\| \\ & + \sup_{0\leq s\leq t}\left\|\int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau)\right\|.\end{aligned}$$

Accordingly one has the following

$$(1 - C_4)\|x_n - x_{n-1}\|_t \leq \sup_{0\leq s\leq t}\left\| \int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau \right\| + \sup_{0\leq s\leq t}\left\| \int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau) \right\|,$$

which gives

$$(1 - C_4)^p\|x_n - x_{n-1}\|_t^p \leq 2^{p-1}\left\{\sup_{0\leq s\leq t}\left\| \int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau \right\|^p + \sup_{0\leq s\leq t}\left\| \int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau) \right\|^p\right\}.$$

Then we arrive at

$$\begin{aligned}(1 - C_4)^p\mathbf{E}\|x_n - x_{n-1}\|_t^p &\leq 2^{p-1}\left\{\mathbf{E}\sup_{0\leq s\leq t}\left\| \int_0^s R(s-\tau)(f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2}))d\tau \right\|^p \right.\\ &\quad \left. + \mathbf{E}\sup_{0\leq s\leq t}\left\| \int_0^s R(s-\tau)(g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2}))dw(\tau) \right\|^p\right\}\\ &\leq 2^{p-1}\left\{\mathbf{E}T^{p-1}M_T^p \sup_{0\leq s\leq t}\int_0^s \|f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2})\|^p d\tau \right.\\ &\quad \left. + c(p,T)\sup_{0\leq s\leq t}\|R(s)\|^p \int_0^t \|g(\tau,\pi_\tau x_{n-1}) - g(\tau,\pi_\tau x_{n-2})\|_\lambda^p d\tau\right\}\\ &\leq 2^{p-1}\left\{\mathbf{E}T^{p-1}M_T^p \int_0^t \|f(\tau,\pi_\tau x_{n-1}) - f(\tau,\pi_\tau x_{n-2})\|^p d\tau \right.\\ &\quad \left. + c(p,T)M_T^p\mathbf{E}\int_0^t \|g(\tau,\pi_\tau x_{n-1} - g(\tau,\pi_\tau x_{n-2}\|_\lambda^p d\tau\right\}\\ &\leq 2^{p-1}\left\{M_T^p T^{p-1}C_1\int_0^t \mathbf{E}\|x_{n-1} - x_{n-2}\|_\tau^p d\tau \right.\\ &\quad \left. + c(p,T)M_T^p C_2\int_0^t \mathbf{E}\|x_{n-1} - x_{n-2}\|_\tau^p d\tau\right\}\\ &\leq 2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)\int_0^t \mathbf{E}\|x_{n-1} - x_{n-2}\|_s^p ds.\end{aligned}$$

As before, we also have

$$\mathbf{E}\|x_n - x_{n-1}\|_t^p \leq \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1 - C_4)^p}\int_0^t \mathbf{E}\|x_{n-1} - x_{n-2}\|_s^p ds.$$

So,

$$\begin{aligned}\mathbf{E}\|x_2 - x_1\|_t^p &\leq \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1 - C_4)^p}\int_0^t \mathbf{E}\|x_1 - x_0\|_s^p ds\\ &\leq \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1 - C_4)^p}\int_0^t \mathbf{E}\|x_1 - x_0\|_t^p ds\\ &\leq \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1 - C_4)^p}\mathbf{E}\|x_1 - x_0\|_t^p t\end{aligned}$$

$$\begin{aligned}
\mathbf{E}\|x_3 - x_2\|_t^p &\le \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\int_0^t \mathbf{E}\|x_2 - x_1\|_s^p ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^2 \int_0^t \mathbf{E}\|x_1 - x_0\|_s^p s ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^2 \mathbf{E}\|x_1 - x_0\|_t^p \int_0^t s ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^2 \frac{t^2}{2}\mathbf{E}\|x_1 - x_0\|_t^p.
\end{aligned}$$

And,

$$\begin{aligned}
\mathbf{E}\|x_4 - x_3\|_t^p &\le \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\int_0^t \mathbf{E}\|x_3 - x_2\|_s^p ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^3 \int_0^t \frac{s^2}{2}\mathbf{E}\|x_1 - x_0\|_s^p ds.
\end{aligned}$$

Inductively, it is possible to write

$$\mathbf{E}\|x_n - x_{n-1}\|_t^p \le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-1} \int_0^t \frac{s^{n-2}}{(n-2)!}\mathbf{E}\|x_1 - x_0\|_s^p ds.$$

Then we have the following inequality

$$\begin{aligned}
\mathbf{E}\|x_{n+1} - x_{n-1}\|_t^p &\le \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\int_0^t \mathbf{E}\|x_n - x_{n-1}\|_s^p ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^n \int_0^t \int_0^s \frac{\tau^{n-2}}{(n-2)!}\mathbf{E}\|x_1 - x_0\|_\tau^p d\tau ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^n \int_0^t \mathbf{E}\|x_1 - x_0\|_s^p \int_0^s \frac{\tau^{n-2}}{(n-2)!} d\tau ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^n \int_0^t \frac{s^{n-1}}{(n-2)!}\mathbf{E}\|x_1 - x_0\|_s^p ds.
\end{aligned}$$

And consequently

$$\begin{aligned}
\mathbf{E}\|x_n - x_{n-1}\|_t^p &\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-1} \int_0^t \frac{s^{n-2}}{(n-2)!}\mathbf{E}\|x_1 - x_0\|_s^p ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-1} \mathbf{E}\|x_1 - x_0\|_t^p \int_0^t \frac{s^{n-2}}{(n-2)!} ds \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-1} \mathbf{E}\|x_1 - x_0\|_t^p \frac{t^{n-1}}{(n-1)!} \\
&\le \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-1} \mathbf{E}\|x_1 - x_0\|_t^p \frac{T^{n-1}}{(n-1)!}\ .
\end{aligned}$$

Thus, for any fixed ϵ_n, we obtain

$$\mathbf{P}\{\|x_n - x_{n-1}\|_t > \epsilon_n\} \le (1/\epsilon_n)^p \mathbf{E}\|x_n - x_{n-1}\|_t^p \quad \text{(Markov's inequality)}$$

Hence, if we set $\epsilon_n = (\frac{1}{n-1})^{1/p}$, then

$$\mathbf{P}\Big\{\|x_n - x_{n-1}\|_t > (\frac{1}{n-1})^{1/p}\Big\} \leq \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-1} \times \mathbf{E}\|x_1 - x_0\|_t^p \frac{T^{n-1}}{(n-2)!}$$
$$\leq \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p} T \times \left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-2} \mathbf{E}\|x_1 - x_0\|_t^p \frac{T^{n-2}}{(n-2)!}.$$

Computing the sum yields

$$\sum_{n\geq 0} \mathbf{P}\Big\{\|x_n - x_{n-1}\|_t > (\frac{1}{n-1})^{1/p}\Big\} \leq \mathbf{E}\|x_1 - x_0\|_t^p \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p} T \times \sum_{n\geq 0}\left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p}\right)^{n-2} \frac{T^{n-2}}{(n-2)!}$$
$$\leq \frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p} T \times \mathbf{E}\|x_1 - x_0\|_t^p \exp\left(\frac{2^{p-1}M_T^p(T^{p-1}C_1 + c(p,T)C_2)}{(1-C_4)^p} T\right) < \infty.$$

According to Borel-Cantelli lemma $\lim_{n\to\infty}\|x_n - x_{n-1}\|_t^p \to 0$, this shows that $\{x_n\}$ is Cauchy in $\mathfrak{B}_T$. Then $x_n(t) \to x(t)$ uniformly in t on [0, T] and $x(t)$ is indeed a unique mild solution satisfying that $x(t) = \varphi(t), t \in [-r, 0]$. ■

4. Exponential Stability in the p-th Moment of the Mild Solution to Equation (1)

In this section, we first consider the exponential stability in p-mean of a trivial solution of equation (1). Assume from now on that $f(t,0) = g(t,0) = h(t,0) \equiv 0$ a.e. t so that equation (1) admits a trivial solution. Let $x(t) = x(t,\varphi)$ be a mild solution of equation (1) where $\varphi(t)$ is the past process.

Definition 4.1. The trivial solution of equation (1) is said to be exponentially stable in p-mean if there exist positive constants K, v such that

$$\mathbf{E}\|x(t;\varphi)\|^p \leq K\,\mathbf{E}\|\varphi\|_0^p \exp(-vt), \quad t \geq 0.$$

The following lemmas are needed in the proves of the main results.

Lemma 4.1. *Let $p > 2$. Suppose that the resolvent operator $(R(t))_{t\geq 0}$ satisfies the following condition: $\|R(t)\| \leq Me^{-at}$ for all $t \geq 0$, where $M \geq 1$ and $a > 0$. Then, for any stochastic process $F : [0,\infty) \to X$ which is strongly measurable with $\int_0^T E\|F(t)\|^p dt < \infty, 0 < T \leq \infty$, the following inequality holds for $0 < t \leq T$:*

$$\mathbf{E}\left\| \int_0^t R(t-s)F(s)ds \right\|^p \le M^p \delta_1 \int_0^t \exp(-a(t-s))\mathbf{E}\|F(s)\|^p ds,$$

where $\delta_1 = a^{1-p}$.

Proof. If $p > 1$, then let $q = p/(p-1)$. By the Hölder inequality we obtain that for $0 \le t \le T$

$$\begin{aligned}\mathbf{E}\left\| \int_0^t R(t-s)F(s)ds \right\|^p &\le \mathbf{E}\Big(\int_0^t Me^{-(a/q)(t-s)}e^{-(a/p)(t-s)}\|F(s)\|ds\Big)^p \\ &\le M^p\mathbf{E}\Big(\int_0^t e^{-a(t-s)}ds\Big)^{1/(p-1)}\Big(\int_0^t e^{-a(t-s)}\|F(s)\|^p ds\Big) \\ &\le M^p a^{1-p}\int_0^t e^{-a(t-s)}\mathbf{E}\|F(s)\|^p ds,\end{aligned}$$

which completes the proof. ■

Lemma 4.2. *Let* $p > 2$. *Suppose that* $\|R(t)\| \le Me^{-at}$ *for all* $t \ge 0$, *where* $M \ge 1$ *and* $a > 0$. *Then, for any* F_t*−adapted predictable process* $: \Phi[0,\infty) \to \sigma(\lambda)(Y,X)$ *with* $\int_0^t E\|\Phi(s)\|_\lambda^2 ds < \infty, t \ge 0,$ *the following inequality holds :*

$$\mathbf{E}\left\| \int_0^t R(t-s)\Phi(s)dw(s) \right\|^p \le M^p \delta_2 \int_0^t \exp(-a(t-s))\mathbf{E}\|\Phi(s)\|_\lambda^p ds,$$

where $\delta_2 = (p(p-1)/2)^{p/2}((p-2)/(2f(p-1)))^{(p/2)-1}$ *with* $0^0 := 1$.

Proof. It is known from Lemma 7.7 of [8] that the next inequality holds.

$$\mathbf{E}\left\| \int_0^t R(t-s)\Phi(s)dw(s) \right\|^p \le \mathcal{C}\Big\{ \int_0^t (\mathbf{E}\|R(t-s)\Phi(s)\|_\lambda^p)^{2/p}ds\Big\}^{p/2},$$

where $\mathcal{C} = (p(p-1)/2)^{p/2}$. Let $q = p/(p-1)$ an

$$\begin{aligned}\mathbf{E}\left\| \int_0^t R(t-s)\Phi(s)dw(s) \right\|^p &\le \mathcal{C}M^p\Big\{ \int_0^t (e^{-ap(t-s)}\mathbf{E}\|\Phi(s)\|_\lambda^p)^{2/p}ds\Big\}^{p/2} \\ &\le \mathcal{C}M^p\Big\{ \int_0^t (e^{-ap(1/q)(t-s)}e^{(-a(t-s))}\mathbf{E}\|\Phi(s)\|_\lambda^p)^{2/p}ds\Big\}^{p/2} \\ &\le \mathcal{C}M^p(2ar/q)^{1-p/2}\int_0^t e^{-a(t-s)}\mathbf{E}\|\Phi(s)\|_\lambda^p ds.\end{aligned}$$

This completes the proof. ■

Using these lemmas we can state the result that follows.

Theorem 4.1. *Let the hypotheses of Theorem 3.1 hold. Then, the mild solution of equation (1) is exponentially stable in p-mean provided that*

$$a > \beta = \beta(a, C_1, C_2, C_4) = \frac{3^{p-1}M^p(\delta_1 C_1 + \delta_2 C_2)}{(1-C_4)^p}. \tag{7}$$

where $\delta_1 = a^{1-p}$ *and* $\delta_2 = (p(p-1)/2)^{p/2}((p-2)/(2a(p-1)))^{(p/2)-1}$.

Proof. Consider the mild solution

$$\begin{aligned} x(s) &= R(s)(\varphi(0)+h(0,\varphi)) - h(t,\pi_s x) + \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \\ &\quad + \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau), a.s., \quad s \in [0,T]. \end{aligned} \tag{8}$$

By assumption (**H2**) while observing that $h(t,0) \equiv 0$ a.e. t, we have

$$\begin{aligned} \|x(s)\| &\leq \|R(s)\|\|\varphi(0)\| + \|R(s)\|\|h(0,\varphi)\| + \|h(s,\pi_s x)\| + \left\| \int_0^s R(s-\tau)f(\tau\pi_\tau x)d\tau \right\| \\ &\quad + \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\| \\ &\leq \|R(s)\|\|\varphi(0)\| + \|R(s)\|\|h(0,\varphi) - h(s,0)\| + \|h(s,\pi_s x) - h(s,0)\| \\ &\quad + \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| + \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\| \\ &\leq \|R(s)\|\|\varphi\|_0 + C_4\|R(s)\|\|\varphi\|_0 + C_4\|x\|_s \\ &\quad + \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| + \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\| \\ &\leq \|R(s)\|\|\varphi\|_0(C_4+1) + C_4\|x\|_t + \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| \\ &\quad + \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\|. \end{aligned}$$

Taking the supremum, we obtain that

$$\begin{aligned} \sup_{0\leq s\leq t} \|x(s)\| &\leq \sup_{0\leq s\leq t} \|R(s)\|\|\varphi\|_0(1+C_4) + C_4 \sup_{0\leq s\leq t} \|x\|_t + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| \\ &\quad + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\| \\ &\leq \sup_{0\leq s\leq t} M\exp(-as)\|\varphi\|_0(1+C_4) + C_4\|x\|_t + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| \\ &\quad + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\| \\ &\leq M\exp(ar)\|\varphi\|_0(1+C_4) + C_4\|x\|_t + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| \\ &\quad + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\| \\ &\leq M\exp(-at)\|\varphi\|_0(1+C_4) + C_4\|x\|_t + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| \\ &\quad + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\|. \end{aligned} \tag{9}$$

According to (9)

$$\begin{aligned} (1-C_4)\|x\|_t \leq M\exp(-at)\|\varphi\|_0(C_4+1) + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau \right\| \\ + \sup_{0\leq s\leq t} \left\| \int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau) \right\|. \end{aligned}$$

Thus, one has

$$\begin{aligned}(1-C_4)^p\|x\|_t^p &\le 3^{p-1}\Big\{M^p\exp(-pat)\|\varphi\|_0^p(1+C_4)^p + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)f(\tau,\pi_\tau x)d\tau\Big\|^p \\ &\quad + \sup_{0\le s\le t}\Big\|\int_0^s R(s-\tau)g(\tau,\pi_\tau x)dw(\tau)\Big\|^p\Big\} \\ &\le 3^{p-1}\Big\{M^p\exp(-at)\|\varphi\|_0^p(1+C_4)^p + \Big\|\int_0^t R(t-\tau)f(\tau,\pi_\tau x)d\tau\Big\|^p \\ &\quad + \Big\|\int_0^t R(t-\tau)g(\tau,\pi_\tau x)dw(\tau)\Big\|^p\Big\}.\end{aligned} \tag{10}$$

By (**H1**), Lemma 4.1, and Lemma 4.2, we obtain that

$$\begin{aligned}(1-C_4)^p\mathbf{E}\|x\|_t^p &\le 3^{p-1}\Big\{M^p\exp(-at)\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + \mathbf{E}\Big\|\int_0^t R(t-s)f(s,\pi_s x)ds\Big\|^p \\ &\quad + \mathbf{E}\Big\|\int_0^t R(t-s)g(s,\pi_s x)dw(s)\Big\|^p\Big\} \\ &\le 3^{p-1}\Big\{M^p\exp(-at)\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + M^p\delta_1\int_0^t \exp(-f(t-s))\mathbf{E}\|f(s,\pi_s x)\|^p ds \\ &\quad + M^p\delta_2\int_0^t \exp(-f(t-s))\mathbf{E}\|g(s,\pi_s x)\|_\lambda^p ds\Big\} \\ &\le 3^{p-1}\{M^p\exp(-at)\mathbf{E}\|\varphi\|_0^p(1+C_4)^p \\ &\quad + M^p\delta_1\int_0^t \exp(-f(t-s))\mathbf{E}\|f(s,\pi_s x)-f(s,0)\|^p ds \\ &\quad + +M^p\delta_2\int_0^t \exp(-f(t-s))\mathbf{E}\|g(s,\pi_s x)-g(s,0)\|_\lambda^p ds\Big\} \\ &\le 3^{p-1}\Big\{M^p\exp(-at)\mathbf{E}\|\varphi\|_0^p(1+C_4)^p + M^p\delta_1 C_1\int_0^t \exp(-f(t-s))\mathbf{E}\|x\|_s^p ds \\ &\quad + M^p\delta_2 C_2\int_0^t \exp(-f(t-s))\mathbf{E}\|x\|_s^p ds\Big\}, \quad (\text{Condition } (\mathbf{H1})) \\ &\le 3^{p-1}\{M^p\exp(-at)\mathbf{E}\|\varphi\|_0^p(1+C_4)^p \\ &\quad + M^p(\delta_1C_1+\delta_2C_2)\int_0^t \exp(-f(t-s))\mathbf{E}\|x\|_s^p ds\Big\}.\end{aligned}$$

This implies that

$$\exp(at)\mathbf{E}\|x\|_t^p \le \frac{3^{p-1}M^p\mathbf{E}\|\varphi\|_0^p(1+C_4)^p}{(1-C_4)^p} + \frac{3^{p-1}M^p(\delta_1C_1+\delta_2C_2)}{(1-C_4)^p}\int_0^t \exp(as)\mathbf{E}\|x\|_s^p ds.$$

By Gronwall's lemma, we have following inequality

$$\exp(at)\mathbf{E}\|x\|_t^2 \le \frac{3^{p-1}M^p(1+C_4)^p}{(1-C_4)^p}\mathbf{E}\|\varphi\|_0^p\exp\Big(\frac{3^{p-1}M^p(\delta_1C_1+\delta_2C_2)}{(1-C_4)^p}t\Big).$$

Consequently, we have

$$\mathbf{E}\|x(t)\|^p \le K\,\mathbf{E}\|\varphi\|_0^p\exp(-\nu t), \quad t\ge 0,$$

where $\nu = a - \beta$ and $K = \frac{3^{p-1}M^p(1+C_4)^p}{(1-C_4)^p}$. ■

5. The Pathwise Stability of the Mild Solution to Equation(1)

Now we state, our main result, the exponential stability of the sample paths of the mild solution of equation (1).

Theorem 5.1. *Suppose that all the conditions of Theorem 4.1 hold. Then the mild solution of equation (1) satisfies*

$$\|x(t)\|^p \leq \delta\, \mathbf{E}\|\varphi\|_0^p \exp(-(a-\beta)t/2), \quad a.s.$$

where $\delta = \exp((a-\beta)/2)$.

Proof. Let N be a sufficiently large positive integer. Let $N \leq t \leq N+1$. Then,

$$\begin{aligned} x(t) &= R(t-N)(x(N) + h(N, x(N))) - h(t, \pi_t x) + \int_N^t R(t-s) f(s, \pi_s x) ds \\ &+ \int_N^t R(t-s) g(s, \pi_s x) dw(s). \end{aligned}$$

So,

$$\begin{aligned} \|x(t)\| \leq{} & \|R(t-N)\| \, \|(x(N) + h(N, x(N)))\| + \|h(t, \pi_t x)\| + \left\| \int_N^t R(t-s) f(s, \pi_s x) ds \right\| \\ & + \left\| \int_N^t R(t-s) g(s, \pi_s x) dw(s) \right\| \\ \leq{} & \|R(t-N)\| \, \|(x(N) + h(N, x(N)))\| + \|h(t, \pi_t x) - h(t, 0)\| + \left\| \int_N^t R(t-s) f(s, \pi_s x) ds \right\| \\ \leq{} & \|R(t-N)\| \, \|(x(N) + h(N, x(N)))\| + C_4 \|x\|_t + \left\| \int_N^t R(t-s) f(s, \pi_s x) ds \right\| \\ & + \left\| \int_N^t R(t-s) g(s, \pi_s x) dw(s) \right\|. \end{aligned} \tag{11}$$

Then,

$$\begin{aligned} (1 - C_4)\|x(t)\| \leq{} & \|R(t-N)\| \, \|(x(N) + h(N, x(N)))\| \\ & + \left\| \int_N^t R(t-s) f(s, \pi_s x) ds \right\| + \left\| \int_N^t R(t-s) g(s, \pi_s x) dw(s) \right\|. \end{aligned}$$

Thus, for any fixed $\mathfrak{B}_N$, we obtain

$$\begin{aligned} \mathbf{P}\left\{ \sup_{N \leq t \leq N+1} \|x(t)\| > \epsilon_N \right\} \leq{} & \mathbf{P}\left\{ \sup_{N \leq t \leq N+1} (1/(1-C_4)) \|R(t-N)\| \, \|x(N) + h(N, x(N))\| > \epsilon_N/3 \right\} \\ & + \mathbf{P}\left\{ \sup_{N \leq t N+1} (1/(1-C_4)) \left\| \int_N^t R(t-s) f(s, \pi_s x) ds \right\| > \epsilon_N/3 \right\} \\ & + \mathbf{P}\left\{ \sup_{N \leq t \leq N+1} (1/(1-C_4)) \left\| \int_N^t R(t-s) g(s, \pi_s x) dw(s) \right\| > \epsilon_N/3 \right\} \end{aligned}$$

$$
\begin{aligned}
&\le (3/\epsilon_N)^p\mathbf{E}\Big(\sup_{N\le t\le N+1}(1/(1-C_4)^p)\,\|R(t-N)\|^p\,\|x(N)+h(N,x(N))\|^p\Big)\\
&\quad+(3/\epsilon_N)^p\mathbf{E}\Big(\sup_{N\le t\le N+1}(1/(1-C_4)^p)\,\Big\|\int_N^t R(t-s)f(s,\pi_s x)ds\Big\|^p\Big)\\
&\quad+(3/\epsilon_N)^p\mathbf{E}\Big(\sup_{N\le t\le N+1}(1/(1-C_4)^p)\,\Big\|\int_N^t R(t-s)g(s,\pi_s x)dw(s)\Big\|^p\Big)\\
&= I_1+I_2+I_3.
\end{aligned}
$$

In view of assumptions (**H1**)- (**H2**), we have that

$$
\begin{aligned}
I_1 &\le (3/\epsilon_N(1-C_4))^p\mathbf{E}\Big(2^p\sup_{N\le t\le N+1}M^pe^{-pat}(\|x(N)\|^p+C_4\|x\|_N^p)\Big)\\
&\le (3/\epsilon_N(1-C_4))^p2^pM^p(C_4+1)\mathbf{E}\|x\|_N^p\\
&\le (3/\epsilon_N(1-C_4))^p2^pM^p(C_4+1)K\,\mathbf{E}\|\varphi\|_0^p\exp(-\nu N),
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le (3/\epsilon_N(1-C_4))^p\sup_{N\le t\le N+1}\int_N^t\mathbf{E}(\|R(t-s)\|\ \ \|f(s,\pi_s x)\|)^pds\\
&\le (3/\epsilon_N(1-C_4))^p\sup_{N\le t\le N+1}\int_N^t M^pe^{-4at}\ \ \mathbf{E}\|f(s,\pi_s x)\|^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^p\int_N^{N+1}\ \mathbf{E}\|f(s,\pi_s x)\|^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^p\int_N^{N+1}\ \mathbf{E}\|f(s,\pi_s x)-f(s,0)\|^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^pC_1\int_N^{N+1}\mathbf{E}\|x\|_s^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^pC_1\int_N^{N+1}K\,\mathbf{E}\|\varphi\|_0^p\exp(-\nu s)ds\\
&\le (3/\epsilon_N(1-C_4))^pM^pC_1K\,\mathbf{E}\|\varphi\|_0^p\exp(-\nu N)\int_N^{N+1}ds\\
&\le (3/\epsilon_N(1-C_4))^pM^pC_1K\,\mathbf{E}\|\varphi\|_0^p\exp(-\nu N).
\end{aligned}
$$

Finally, by invoking Lemma 2.5 together with assumption (**H1**), we get

$$
\begin{aligned}
I_3 &\le (3/\epsilon_N(1-C_4))^pM^pk_p\int_N^{N+1}\ \mathbf{E}\|g(s,\pi_s x)\|_\lambda^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^pk_p\int_N^{N+1}\ \mathbf{E}\|g(s,\pi_s x)-g(s,0)\|_\lambda^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^pk_pC_2\int_N^{N+1}\mathbf{E}\|x\|_s^pds\\
&\le (3/\epsilon_N(1-C_4))^pM^pk_pC_2K\,\mathbf{E}\|\varphi\|_0^p\exp(-\nu N).
\end{aligned}
$$

Using these estimates, it follows that

$$
\begin{aligned}
\mathbf{P}\left\{\sup_{N\le t\le N+1}\|x(t)\|>\epsilon_N\right\} &\le (3/\epsilon_N(1-C_4))^p 2^p M^p (C_4+1)K\,\mathbf{E}\|\varphi\|_0^p \exp(-\nu N)\\
&\quad + (3/\epsilon_N(1-C_4))^p M^p C_1 K\,\mathbf{E}\|\varphi\|_0^p \exp(-\nu N)\\
&\quad + (3/\epsilon_N(1-C_4))^p M^p k_p C_2 K\,\mathbf{E}\|\varphi\|_0^p \exp(-\nu N)\\
&\le (3/\epsilon_N(1-C_4))^p K M^p (2^p(C_4+1)+C_1\\
&\quad + k_p C_2)\mathbf{E}\|\varphi\|_0^p \exp(-\nu N).
\end{aligned}
$$

Hence, if we set $\epsilon_N = (\mathbf{E}\|\varphi\|_0^p)^{1/p}\exp(-\nu N/2p)$, then

$$
\begin{aligned}
\mathbf{P}\left\{\sup_{N\le t\le N+1}\|x(t)\|>(\mathbf{E}\|\varphi\|_0^p)^{1/p}\exp(-\nu N/2p)\right\} &\le (3/(1-C_4))^p K M^p (2^p(C_4+1)+C_1+k_pC_2)\\
&\quad \times (\mathbf{E}\|\varphi\|_0^p)^{-1}\exp(\nu N/2)\mathbf{E}\|\varphi\|_0^p\exp(-\nu N)\\
&\le (3/(1-C_4))^p K M^p (2^p(C_4+1)\\
&\quad + C_1+k_pC_2)\exp(-\nu N/2).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\sum_{N\ge 0}\mathbf{P}\left\{\sup_{N\le t\le N+1}\|x(t)\|>(\mathbf{E}\|\varphi\|_0^p)^{1/p}\exp(-\nu N/2p)\right\}\\
&\le (3/(1-C_4))^p K M^p (2^p(C_4+1)+C_1+k_pC_2)\sum_{N\ge 0}\exp(-\nu N/2)\\
&\le (3/(1-C_4))^p K M^p (2^p(C_4+1)+C_1\\
&+ k_pC_2)\frac{1}{1-\exp(-\nu/2)}\\
&< \infty.
\end{aligned}
$$

Therefore, by the Borel-Cantelli lemma, one has that

$$
\begin{aligned}
\|x(t)\|^p &\le \mathbf{E}\|\varphi\|_0^p\exp(-\nu N/2)\\
&\le \exp(\nu/2)\mathbf{E}\|\varphi\|_0^p\exp(-\nu(N+1)/2)\\
&\le \exp(\nu/2)\mathbf{E}\|\varphi\|_0^p\exp(-\nu t/2)\\
&\le \exp(\nu/2)\mathbf{E}\|\varphi\|_0^p\exp(-(a-\beta)t/2).
\end{aligned}
$$

Here the proof ends. ■

Remark 5.1. All our results are established for $p>2$. In fact, the case $p=2$, the stochastic convolution inequality is an open question. Even for contraction semigroup, we cannot apply directly Itô formula to establish the inequality for resolvent operator thus we confine ourselves to the case $p>2$.

6. Illustrative Example

In this section, we illustrate the abstract results proved in the previous sections by analyzing an example. Consider the stochastic partial neutral functional integrodifferential equation with finite delays r_1, r_2 and $r_3 (r > r_i \ge 0, i = 1,2,3)$:

$$\begin{cases} d\left\{z(t,\xi)+\alpha_3\int_{-r_3}^{0} z(t+u,\xi)du\right\} \\ \quad = \frac{\partial^2}{\partial x^2}\left\{z(t,\xi)+\alpha_3\int_{-r_3}^{0} z(t+u,\xi)du\right\}dt \\ \quad + \int_0^t b(t-s)\frac{\partial^2}{\partial x^2}\left\{z(s,\xi)+\alpha_3\int_{-r_3}^{0} z(s+u,\xi)du\right\}dsdt \\ \quad + \alpha_1\int_{-r_1}^{0} z(t+u,\xi)dudt+\alpha_2 z(t-r_2,\xi)d\beta(t), \quad t\geq 0, \alpha_i\geq 0, i=1,2,3; \\ z(t,0)=z(t,\pi)=0, \quad t>0; \\ z(t,\xi)=z_0(t,\xi), \quad t\in[-r,0], \xi\in[0,\pi]; \end{cases} \tag{12}$$

where $\beta(t)$ denotes the standard $\mathbb{R}$-valued Brownian motion, $g:\mathbb{R}^+\to\mathbb{R}$. a continuous function.
Let $X=L^2(0,\pi)$ with the norm$\|.\|$ and $e_n:=\sqrt{2/\pi}\,\sin(nx), n=1,2,3,\ldots,$ denote the completed orthonormal basis in X. Let $U=\mathbb{R}, Q=1, \lambda_1=1, \lambda_n=0, n>1$. Let $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space where the Brownian motion is defined and set for $t\in[-r,0], \mathcal{F}_t=\mathcal{F}_0$. Define $A:\mathcal{D}(A)\subset X\to X$ by $A=\partial^2/\partial x^2$, with domain $\mathcal{D}(A)=H^2(0,\pi)\cap H_0^1(0,\pi)$. Then

$$Av=-\sum_{n=1}^{\infty} n^2 < v,e_n > e_n, \quad v\in\mathcal{D}(A),$$

where e_n is the orthonormal set of eigenvectors of A.

It is well known that A is the infinitesimal generator of an C_0- semigroup $\{S(t), t\geq 0\}$ in X. Let $B:\mathcal{D}(A)\subset X\to X$ be the operator defined by $B(t)z=b(t)Az$ for $t\geq 0$ and $z\in\mathcal{D}(A)$. Now, we define

$$\begin{cases} h(t,\pi_t x)=\alpha_3\int_{-r_3}^{0} x(t+u)du, \\ f(t,\pi_t x)=\alpha_1\int_{-r_1}^{0} x(t+u)du, \\ g(t,\pi_t x)=\alpha_2 x(t-r_2). \end{cases}$$

If we put

$$\begin{cases} x(t)=z(t,.), \quad t\geq 0 \\ \varphi(t)=z_0(t,.) \quad t\in[-r,0], \end{cases}$$

then equation (12) takes the following abstract form

$$\begin{cases} d[x(t)+h(t,\pi_t x)] = A[x(t)+h(t,\pi_t x)] + \int_0^t B(t-s)[x(s)+h(s,\pi_s x)]ds + f(t,\pi_t x))dt \\ \qquad\qquad + g(t,\pi_t x)dw(t), \quad t>0, \\ x(t)=\varphi(t), \quad t\in[-r,0]. \end{cases} \tag{13}$$

We assume that b is bounded uniformly continuous on $\mathbb{R}_+$. Moreover, if b is bounded and $\mathcal{C}^1$ function such that b' is bounded and uniformly continuous, then (R1) and (R2) are satisfied, and hence, by Theorem 2.1, system (2) has a resolvent operator $(R(t))_{t\geq 0}$ on X.

Using Lemma 5.2 of [7], let $\mu > \delta > 1$ and $b(t) < \frac{1}{a}\exp(-\beta)$, for all $t \geq 0$. Then the resolvent operator of equation (1) decays exponentially to zero. Specifically $\|R(t)\| < \exp(-at)$ where $a = 1 - 1/\delta$.
Next, we have

$$\|h(t,\pi_t x\| \leq \alpha_3 \int_{-r_3}^{0} \|x\|_t du \leq \alpha_3 r_3 \|x\|_t$$

And for $p > 2$, we have

$$\|f(t,\pi_t x)\|^p \leq \left\| \alpha_1 \int_{-r_1}^{0} x(t+u)du \right\|^p \leq \alpha_1^p \left(\int_{-r_1}^{0} \|x\|_t du \right)^p \leq \alpha_1^p r_1^p \|x\|_t^p,$$

$$\|g(t,\pi_t x)\|_\lambda^p \leq \alpha_2^p \|x(t-r_2)\|^p \leq \alpha_2^p \|x\|_t^p,$$

$$\|f(t,\pi_t x)\|^p + \|g(t,\pi_t x)\|_\lambda^p \leq (\alpha_1^p r_1^p + \alpha_2^p)\|x\|_t^p \leq (\alpha_1^p r_1^p + \alpha_2^p)(1 + \|x\|_t^p).$$

Therefore, by Theorem 3.1, equation (12) has a unique mild solution. Moreover, by Theorem 4.1, the solution is also exponentially stable in p-mean provided $a > \beta$, where

$$\beta = \frac{3^{p-1}(\alpha_1^p r_1^p a^{1-p} + \alpha_2^p (p(p-1)/2)^{p/2}((p-2)/(2a(p-1)))^{(p/2)})}{(1-\alpha_3 r_3)^p},$$

and lastly by Theorem 5.1,

$$\|x(t)\|^p \leq \exp((a-\beta)/2)\mathbf{E}\|\varphi\|_0^p \exp(-(a-\beta)t/2), \quad a.s.$$

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References

[1] T. Caraballo, and J. Real, Partial differential equations with delayed random perturbations: Existence, uniqueness and stability of solutions, *Stochastic Analysis and Applications* **11**, (1993), 497-511.

[2] T. Caraballo, Asymptotic exponential stability of stochastic partial differential equations with delays, *Stochastics* **33**, (1990), 27-47.

[3] J. Duan, and L. Wan, Exponential stability of non-autonomous stochastic partial differential equations with finite memory, *Statistics and Probability Letters* **78**(5), (2008), 490-498.

[4] M. Dieye, M. A. Diop, and K. Ezzinbi, Controllability for some integrodifferential equations driven by vector measures, *Mathematical Methods in the Applied Sciences* **40**(6), (2017), 2090-2106.

[5] M. Dieye, M. A. Diop, and K. Ezzinbi, Necessary conditions of optimality for some stochastic

integrodifferential equations of neutral type on Hilbert spaces, *Applied Mathematics and Optimization* **77**(2), (2016), 343-375.

[6] M. A. Diop, K. Ezzinbi, and M. Lo, Existence and uniqueness of mild solutions to some neutral stochastic partial functional integrodifferential equations with non- Lipschitz coefficients, *International Journal of Mathematics and Mathematical Sciences* **2012**, (2012), ID 748590.

[7] M. Dieye, M. A. Diop, and K. Ezzinbi, On exponential stability of mild solutions for some stochastics partial integrodifferential equations, *Statistics and Probability Letters* **123**, (2017), 61-76.

[8] G. Da Prato, and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[9] K. Ezzinbi, and S. Ghnimi, Existence and regularity of solutions for neutral partial functional integrodifferential equations, *Nonlinear Analysis: Real World Applications* **11**, (2010), 2335- 2344.

[10] T. Funaki, Random motion of string and related stochastic evolution equations, *Nagoya Mathematical Journal* **89**, (1983), 129-193.

[11] R. C. Grimmer, Resolvent opeators for integral equations in a Banach space, *Transactions of the American Mathematical Society* **273**, (1982), 333-349.

[12] T. E. Govindan, Almost sure exponential stability for stochastic neutral partial functional differential equations, *Stochastics* **77**, (2005), 139-154.

[13] R. C. Grimmer, and A. J. Pritchard, Analytic resolvent operators for integral equations in Banach space, *Journal of Differential Equations* **50**(2), (1983), 234-259.

[14] T. E. Govindan, Existence and stability of solutions of stochastic semilinear functional differential equation, *Stochastic Analysis and Applications* **20**(6), (2007), 1257-1280.

[15] E. Hernandez, and H. R. Henriquez, Existence results for partial neutral functional differential equations with unbounded delay, *Jounnal of Mathematical Analysis and Applications* **221**, (1998), 452- 475.

[16] U. G. Haussmann, Asymptotic stability of the linear Itô equation in infinite dimensions, *Jounnal of Mathematical Analysis and Applications* **65**, (1978), 219-235.

[17] A. Ichikawa, Stability of semilinear stochastic evolution equations, J*ounnal of Mathematical Analysis and Applications* **90**, (1982), 12-44.

[18] K. Lui, and A. Truman, A note on almost sure exponential stability for stochastic partial functional differential equations, *Statistics and Probability Letters* **50**, (2002), 273-278.

[19] J. Luo, Exponential stability for stochastic neutral partial functional differential equations, *Jounnal of Mathematical Analysis and Applications* **355**, (2009), 414-425.

[20] X. Mao, Exponential stability for stochastic delay equations in Hilbert spaces, *Chinese Quarterly Journal of Mathematical Analysis and Applications* **42**, (1991), 77- 85.

[21] A. E. Rodkina, On existence and uniqueness of solution of stochastic differential equations with heredity, *Stochastics* **12**(3-4), (1984), 187- 200.

[22] T. Taniguchi, Almost sure exponential stability for stochastic partial functional differential equations, *Stochastic Analysis and Applications* **16**, (1998), 965-975.

[23] T. Taniguchi, Asymptotic stability theorems of semilinear stochastic evolution equations in Hilbert spaces, *Stochastics and Stochastics Reports* **53**, (1995), 41- 52.

[24] X. Zhang,and D. Ruan, Exponential stability for neutral stochastic functional partial differential equations driven by Brownian motion and fractional Brownian motion, *Journal of Inequalities and Applications* **2018**, (2018), No. 201.

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