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Reflected Discontinuous Backward Doubly Stochastic Differential Equations With Poisson Jumps

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Abstract. *In this paper we prove the existence of a solution for reflected backward doubly stochastic differential equations with poisson jumps (RBDSDEPs) with one continuous barrier where the generator is continuous and also we study the RBDSDEPs with a linear growth condition and left continuity in y on the generator. By a comparison theorem established here for this type of equation we provide a minimal or a maximal solution to RBDSDEPs.*

Key words : Reflected Backward Doubly Stochastic Differential Equations, Random Poisson Measure, Minimal Solution, Comparison Theorem, Discontinuous Generator.

AMS Subject Classifications : 60H05, 60H10, 60H30

1. Introduction

A new kind of backward stochastic differential equations was introduced by Pardoux and Peng [11] in 1994, which is a class of backward doubly stochastic differential equations (BDSDEs for short)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s,\ 0 \leq t \leq T,$$

where ξ is a random variable termed the terminal condition, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are two jointly measurable processes, W and B are two mutually independent standard Brownian motion, with values, respectively in $\mathbb{R}^d$ and $\mathbb{R}$. Several authors interested in weakening this assumption see Bahlali et al [3], Boufoussi et al. [5], Lin. Q [8] and [9], N'zi el al. [10], Shi et al. [13], Wu et al. [15], Zhu et al. [17]. A class of backward doubly stochastic differential equations with jumps was study by Sun el al. [14], Zhu et al. [16] They have proved the existence and uniqueness of solutions for this type of BDSDEs under uniformly Lipschitz conditions.
In addition, Bahlali et al [2] prove the existence and uniqueness of solutions to reflected

backward doubly stochastic differential equations (RBDSDEs) with one continuous barrier and uniformly Lipschitz coefficients. The existence of a maximal and a minimal solution for RBDSDEs with continuous generator is also established.

In this paper, we study the now well-know reflected backward doubly stochastic differential equations with jumps (RBDSDEPs for short):

$$Y_t = \xi + \int_t^T f(s,\Lambda_s)ds + \int_t^T g(s,\Lambda_s)d\overleftarrow{B}_s + \int_t^T dK_s - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds,de),$$
$$0 \le t \le T, \quad (1)$$

where $\Lambda_s = (Y_s, Z_s, U_s)$.

Motivated by the above results and by the result introduced by Fan. X, Ren. Y [6] and Zhu, Q., Shi, Y [16, 17], we establish firstly the existence of the solution of the reflected BDSDE with Poisson jumps (RBDSDEP in short) under the continuous coefficient, also we prove the existence solution of a RBDSDEP where the coefficient f satisfy a linear growth and left continuity in y conditions on the generator of this type of equation.$\mathcal{F}$

The organization of the paper is as follows. In section 2, we give some preliminaires and we consider the spaces of processus also we define the Itô's formula. In section 3, we proof a comparison theorem, section 4 under a continuous conditions on f we obtain the existence of a minimal solution of RBDSDEP, and finally in section 5, we study RBDSDEP where the generator f satisfied a left continuity in y and linear growth conditions.

2. Notations, Assumptions and Definitions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, We suppose that $(\mathcal{F}_t)_{t\geq 0}$ is generated by the following three mutually independent processes:

(i) Let $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$ be two standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively.

(ii) Let random Poisson measure μ on $E \times \mathbb{R}_+$ with compensator $\nu(dt,de) = \lambda(de)dt$, where the space $E = \mathbb{R} - \{0\}$ is equipped with its Borel field $\mathcal{E}$ such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}$ is a martingale for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is a σ finite measure on $\mathcal{E}$ and satisfies $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$.

Let $\mathcal{F}_t^W := \sigma(W_s; 0 \le s \le t)$, $\mathcal{F}_t^\mu := \sigma(\mu_s; 0 \le s \le t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \le s \le T)$, completed with P-null sets. We put, $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\mu$. It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub σ –fields, and hence it is not a filtration.

For $d \in \mathbb{N}^*$, $|\cdot|$ stands for the Euclidian norm in $\mathbb{R}^d \times [0,T]$.

We consider the following spaces of processes:

- We denote by $S^2(0,T,\mathbb{R}^d)$, the set of continuous $\mathcal{F}_t$ –measurable processes $\{\varphi_t; t \in [0,T]\}$, which satisfy $\mathsf{E}(\sup_{0\le t\le T}|\varphi_t|^2) < \infty$.
- Let $\boldsymbol{M}^2(0,T,\mathbb{R}^d)$ denote the set of $d-$ dimensional, $\mathcal{F}_t$ –measurable processes $\{\varphi_t; t \in [0,T]\}$, such that $\mathsf{E}\int_0^T |\varphi_t|^2 dt < \infty$.
- A^2 set of continuous, increasing, $\mathcal{F}_t$ –measurable process $K : [0,T] \times \Omega \to [0,+\infty$ (with $K_0 = 0$, $\mathsf{E}(K_T)^2 < +\infty)]$.
- L^2 set of $\mathcal{F}_T$- measurable random variables $\xi : \Omega \to \mathbb{R}$ with $\mathsf{E}|\xi|^2 < +\infty$.
- We denote by $\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}^d)$, the space of mappings $U : \Omega \times [0,T] \times E \to \mathbb{R}^d$ which are $\wp \otimes \mathcal{E}$ measurable such that

$\|U_t\|^2_{\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}^d)} = \mathsf{E}\int_0^T \|U_t\|^2_{L^2(E,\mathsf{E},\lambda,\mathbb{R}^d)}dt < \infty,$

where $\wp \otimes \mathcal{E}$ denoted the σ –algebra of $\mathcal{F}_t$ –predectable sets of $\Omega \times [0,T]$ and

$\|U_t\|^2_{L^2(E,\mathsf{E},\lambda,\mathbb{R}^d)} = \int_E^T |U_t(e)|^2\lambda(de).$

- Notice also the space $D^2(\mathbb{R}) = S^2(0,T,\mathsf{R}) \times M^2(0,T,\mathsf{R}^d) \times \mathcal{L}^2(0,T,\tilde{\mu},\mathsf{R}) \times A^2$ endowed with the norm

$\|(Y,Z,U,K)\|_{D^2} = \|Y\|_{S^2} + \|Z\|_{M^2} + \|U\|_{\mathcal{L}^2} + \|K\|_{A^2}.$

is a Banach space.

Definition 2.1. A solution of a reflected BDSDEPs is a quadruple of processes (Y,Z,K,U) wich satisfies

$$\begin{cases} i)\ Y \in S^2(0,T,\mathbb{R}),\ Z \in M^2(0,T,\mathbb{R}^d)\ ,K \in A^2\ ,U \in \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}), \\ ii)\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds + \int_t^T g(s,Y_s,Z_s,U_s)d\overleftarrow{B}_s \\ +\int_t^T dK_s - \int_t^T Z_s dW_s - \int_t^T\int_E U_s(e)\tilde{\mu}(ds,de),\ 0 \le t \le T, \\ iii)\ S_t \le Y_t, \quad 0 \le t \le T \quad \text{and} \quad \int_0^T (Y_t - S_t)dK_t = 0. \end{cases}$$

We give the following (**H**) assumptions on the data $\left(\xi,\ f,\ g,\ S\right)$:

(**H.1**) $f:\ [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}) \to \mathbb{R}$;
$g\ :\ [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}) \to \mathbb{R}$ be jointly measurable such that for any $(y,z,u) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})$

$f(\cdot,\omega,y,z,u) \in M^2(0,T,\mathbb{R}),$

$g(\cdot,\omega,y,z,u) \in M^2(0,T,\mathbb{R}).$

(**H.2**) There exist constant $C > 0$ and a constant $0 < \alpha < 1$ such that for every $(\omega,t) \in \Omega \times [0,T]$ and $(y,y') \in \mathbb{R}^2,\ (z,z') \in (\mathbb{R}^d)^2,\ (u,u') \in \left(\mathsf{L}^2(0,T,\tilde{\mu},\mathbb{R})\right)^2$

$$\begin{cases} (i)\ |f(t,\omega,y,z,u) - f(t,\omega,y',z',u')|^2 \le C\left[|y-y'|^2 + |z-z'|^2 + |u-u'|^2\right], \\ (ii)\ |g(t,\omega,y,z,u) - g(t,\omega,y',z',u')|^2 \le C|y-y'|^2 + \alpha\left(|z-z'|^2 + |u-u'|^2\right). \end{cases}$$

(**H.3**) The terminal value ξ be a given random variable in L^2.

(**H.4**) $(S_t)_{t\ge 0}$, is a continuous progressively measurable real valued process satisfying

$$\mathsf{E}\left(\sup_{0\leq t\leq T}(S_t^+)^2\right) < +\infty, \quad \text{where} \quad S_t^+ := \max(S_t, 0).$$

(H.5) $S_T \leq \xi$, P-almost surely.

Theorem [6] **2.1**. *Assume that* $(\mathbf{H.1})-(\mathbf{H.5})$ *hold. Then equation* (1) *admits a unique solution* $(Y,Z,U,K) \in D^2(\mathbb{R})$.

The result depends on the following extension of the well-krown Itô's formula. Its proof follows the same way as Lemma 1.3 of [11].

Lemma 2.1. *Let* $\alpha \in S^2(0,T,\mathbb{R}^k)$, $(\beta,\gamma) \in \left(M^2(\mathbb{R}^k)\right)^2$, $\eta \in M^2(\mathbb{R}^{k\times d})$ *and* $\sigma \in \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}^k)$ *such that:*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \eta_s dW_s + \int_0^t dK_s + \int_0^t\int_E \sigma_s(e)\tilde{\mu}(ds,de),$$

then (i)

$$\begin{aligned}|\alpha_t|^2 = |\alpha_0|^2 &+ 2\int_0^t\langle\alpha_s,\beta_s\rangle ds + 2\int_0^t\langle\alpha_s,\gamma_s\rangle dB_s + 2\int_0^t\langle\alpha_s,\eta_s\rangle dW_s + 2\int_0^t\langle\alpha_s,dK_s\rangle \\ &+ 2\int_0^t\int_E\langle\alpha_{s-},\sigma(e)\tilde{\mu}(ds,de)\rangle - \int_0^t|\gamma_s|^2 ds + \int_0^t|\eta_s|^2 ds + \int_0^t\int_E|\sigma_s(e)|^2\lambda(de)ds \\ &+ \sum_{0\leq t\leq T}(\Delta\alpha_s)^2,\end{aligned}$$

(ii)

$$\begin{aligned}\mathsf{E}|\alpha_t|^2 + \mathsf{E}\int_t^T|\eta_s|^2 ds + \mathsf{E}\int_t^T\int_E|\sigma_s(e)|^2\lambda(de)ds \leq \mathsf{E}|\alpha_T|^2 + 2\mathsf{E}\int_t^T\langle\alpha_s,\beta_s\rangle ds + 2\mathsf{E}\int_t^T\langle\alpha_s,dK_s\rangle \\ + \mathsf{E}\int_t^T|\gamma_s|^2 ds.\end{aligned}$$

3. A Comparison Theorem

Given two parameters (ξ^1,f^1,g,T) and (ξ^2,f^2,g,T), we considere the reflected BDSDEPs, $i = 1,2,$

$$\begin{aligned}Y_t^i = \xi^i &+ \int_t^T f^i(s,Y_S^i,Z_S^i,U_S^i)ds + \int_t^T g(s,Y_S^i,Z_S^i,U_S^i)d\overleftarrow{B}_S \\ &+ \int_t^T dK_S^i - \int_t^T Z_S^i dW_S - \int_t^T\int_E U_S^i(e)\tilde{\mu}(ds,de),\ 0 \leq t \leq T. \qquad (2)\end{aligned}$$

Theorem 3.1. *Assume that the reflected BDSDEP associated with dates* (ξ^1,f^1,g,T), $\left(\textit{resp}\left(\xi^2,f^2,g,T\right)\right)$ *has a solution* $(Y_t^1,Z_t^1,K_t^1,U_t^1)_{t\in[0,T)]}$, (*resp* $(Y_t^2,Z_t^2,K_t^2,U_t^2)_{t\in[0,T]}$). *Each one satisfying the assumption (H). Assume moreover that:*

$$\begin{cases} \xi^1 \le \xi^2, \\ \forall t \le T,\ S_t^1 \le S_t^2, \\ f^1(t, Y_t, Z_t, U_t) \le f^2(t, Y_t, Z_t, U_t). \end{cases}$$

Then we have $\mathsf{P} - a.s.$,

$Y_t^1 \le Y_t^2$.

Proof. Let us show that $\left(Y_t^1 - Y_t^2\right)^+ = 0,$ using the equation (2), we get

$$\begin{aligned} \bar{Y}_t &= Y_t^1 - Y_t^2 \\ &= \bar{\xi} + \int_t^T (f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2))ds + \int_t^T (g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2))d\overleftarrow{B}_s \\ &\quad + \int_t^T (dK_s^1 - dK_s^2) - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_E \bar{U}_s(e)\lambda(ds, de), \end{aligned}$$

where $\bar{\xi} = \xi^1 - \xi^2,\ \bar{Z} = Z^1 - Z^2$ and $\bar{U} = U^1 - U^2$.

Since $\int_t^T (\bar{Y}_s)^+ (g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2))d\overleftarrow{B}_s$ and $\int_t^T (\bar{Y}_s)^+ \bar{Z}_s dW_s$ are a uniformly integrable martingale then taking expectation, we get by applying lemma 2.1

$$\begin{aligned} &\mathsf{E}|(\bar{Y}_t)^+|^2 + \mathsf{E}\int_t^T 1_{\{\bar{Y}_s > 0\}} \|\bar{Z}_s\|^2 ds + \mathsf{E}\int_t^T \int_E 1_{\{\bar{Y}_s > 0\}} |\bar{U}_s(e)|^2 \lambda(de)ds \\ &\le \mathsf{E}|(\bar{\xi})^+|^2 + 2\mathsf{E}\int_t^T (\bar{Y}_s)^+ \left(f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)\right)ds \\ &\quad + 2\mathsf{E}\int_t^T (\bar{Y}_s)^+ (dK_s^1 - dK_s^2) + \mathsf{E}\int_t^T 1_{\{\bar{Y}_s > 0\}} \|g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)\|^2 ds. \end{aligned}$$

Since

$$\begin{cases} (\xi^1 - \xi^2)^+ = 0, \\ \\ \int_t^T (\bar{Y}_s)^+ (dK_s^1 - dK_s^2) = -\int_t^T (Y_s^1 - Y_s^2)^+ dK_s^2 \le 0, \end{cases}$$

we get

$$\begin{aligned} &\mathsf{E}\left\{|(\bar{Y}_t)^+|^2 + \int_t^T 1_{\{\bar{Y}_s > 0\}} \|\bar{Z}_s\|^2 ds + \int_t^T \int_E 1_{\{\bar{Y}_s > 0\}} |\bar{U}_s(e)|^2 \tilde{\mu}(de)ds\right\} \\ &\le 2\mathsf{E}\int_t^T (\bar{Y}_s)^+ (f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2))ds \\ &+ \mathsf{E}\int_t^T 1_{\{\bar{Y}_s > 0\}} \|g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)\|^2 ds, \end{aligned}$$

to obtain, by hypothesis $(H.2)$ and Young's inequality, the following inequality

$$\begin{aligned} &2\mathsf{E}\int_t^T (\bar{Y}_s)^+ \left(f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)\right)ds \\ &\le (2C + 2C^2\epsilon)\mathsf{E}\int_t^T \left|\bar{Y}_s^+\right|^2 ds + \epsilon^{-1}\mathsf{E}\int_t^T \left(|\bar{Z}_s|^2 + \int_E |\bar{U}_s|^2 \lambda(de)\right)ds. \end{aligned}$$

Also, we apply assumption $(H.2)$ for g to arrive at

$$\|g(s,Y_s^1,Z_s^1,U_s^1)-g(s,Y_s^2,Z_s^2,U_s^2)\|^2 \le C|\bar{Y}_s|^2 ds+\alpha\left\{|\bar{Z}_s|^2+|\bar{U}_s|^2\right\}.$$

Then, we have the following inequality

$$\begin{aligned}
&\mathsf{E}\left\{|(\bar{Y}_t)^+|^2+\int_t^T 1_{\{\bar{Y}_s>0\}}\|\bar{Z}_t\|^2 ds+\int_t^T\int_E 1_{\{\bar{Y}_s>0\}}|\bar{U}_s(e)|^2\tilde{\mu}(de)ds\right\}\\
&\le (2C+2C^2\epsilon)\mathsf{E}\int_t^T|\bar{Y}_s^+|^2 ds+\epsilon^{-1}\mathsf{E}\int_t^T\left(|\bar{Z}_s|^2+\int_E|\bar{U}_s|^2\lambda(de)\right)ds\\
&\quad + C\mathsf{E}\int_t^T 1_{\{\bar{Y}_s>0\}}|\bar{Y}_s|^2 ds+\alpha\mathsf{E}\int_t^T\left\{1_{\{\bar{Y}_s>0\}}|\bar{Z}_s|^2+\int_E 1_{\{\bar{Y}_s>0\}}|\bar{U}_s|^2\lambda(de)\right\}ds,\\
&\le (2C^2\epsilon+3C)\mathsf{E}\int_t^T\left|\bar{Y}_s^+\right|^2 ds+(\epsilon^{-1}+\alpha)\mathsf{E}\left\{\int_t^T 1_{\{\bar{Y}_s>0\}}|\bar{Z}_s|^2 ds+\int_t^T\int_E 1_{\{\bar{Y}_s>0\}}|\bar{U}_s|^2\lambda(de)ds\right\}.
\end{aligned}$$

By choosing ϵ such that $0<\epsilon^{-1}+\alpha\le 1$, we have

$$\mathsf{E}|(\bar{Y}_t)^+|^2 \le (3C+2C^2\epsilon)\mathsf{E}\int_t^T\left|\bar{Y}_s^+\right|^2 ds.$$

Then using Gronwall's lemma implies that

$$\mathsf{E}\left[|(\bar{Y}_t)^+|^2\right]=0.$$

Finally, we have

$Y_t^1 \le Y_t^2$. ■

4. Reflected BDSDEPs With Continuous Coefficients

In this section we are interested in weakening the conditions on f. We assume that f and g satisfy the following assumptions:

(**H.6**) There exists $0<\alpha<1$ and $C>0$ $s.t.$ for all (t,ω,y,z,u) $\in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathsf{E},\lambda,\mathbb{R})$,

$$\begin{cases}|f(t,\omega,y,z,u)|\le C(1+|y|+|z|+|u|),\\ |g(t,\omega,y,z,u)-g(t,\omega,y',z',u')|^2\le C|y-y'|^2+\alpha\left\{|z-z'|^2+|u-u'|^2\right\}.\end{cases}$$

(**H.7**) For fixed ω and t, $f(t,\omega,\cdot,\cdot,\cdot)$ is continuous.

The next three lemmas will be useful in the sequel. Before stating them, we recall the following classical lemma, that can be proved by adapting the proof given by J. J. Alibert and K. Bahlali in [1].

Lemma 4.1. Let $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathsf{L}^2(E,\mathsf{E},\lambda,\mathbb{R})\to\mathbb{R}$ be a mesurable function such that:

Lemma 1.

2. *For a.s. every* $(t,\omega)\in[0,T]\times\Omega$, $f(t,\omega,y,z,u)$ *is a continuous.*
3. *There exists a constant* $C>0$ *such that for every* $(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathsf{E},\lambda,\mathbb{R})$, $|f(t,\omega,y,z,u)|\le C(1+|y|+|z|+|u|)$.

Then exists the sequence of functions f_n

$$f_n(t,\omega,y,z,u) = \inf_{\left(y',z',u'\right)\in B^2(\mathbb{R})} \left[f\left(t,\omega,y',z',u'\right) + n\left(\left|y-y'\right| + |z-z'| + \left|u-u'\right|\right)\right],$$

is well defined for each $n \geq C$, and it satisfies, $dP \times dt - a.s.$
(i) Linear growth: $\forall n \geq 1,\ (y,z,u) \in \mathbb{R}\times\mathbb{R}^d\times\mathcal{L}^2,\ |f_n(t,\omega,y,z,u)| \leq C(1+|y|+|z|+|u|)$.
(ii) Monotonicity in n : $\forall\ y,z,u,\ f_n(t,\omega,y,z,u)$ is increases in n.
(iii) Convergence: $\forall(t,\omega,y,z,u) \in [0,T]\times\Omega\times B^2(\mathbb{R})$, if $(t,\omega,y_n,z_n,u_n) \to (t,\omega,y,z,u)$, then $f_n(t,\omega,y_n,z_n,u_n) \to f(t,\omega,y,z,u)$.
(iv) Lipschitz condition: $\forall n \geq 1$, $(t,\omega) \in [0,T]\times\Omega$, $\forall(y,z,u) \in B^2(\mathbb{R})$ and $\left(y',z',u'\right) \in B^2(\mathbb{R})$, we have

$$\left|f_n(t,\omega,y,z,u) - f_n\left(t,\omega,y',z',u'\right)\right| \leq n\left|y-y'\right| + |z-z'| + \left|u-u'\right|.$$

Now given $\xi \in \mathrm{L}^2$, $n \in N$, we consider (Y^n,Z^n,K^n,U^n) and $\left(\text{resp } (V,N,K,M)\right)$ be solutions of the following reflected BDSDEPs:

$$\begin{cases} Y_t^n = \xi + \int_t^T f_n(s,Y_s^n,Z_s^n,U_s^n)ds + \int_t^T g(s,Y_s^n,Z_s^n,U_s^n)d\overleftarrow{B}_s \\ + \int_t^T dK_s^n - \int_t^T Z_s^n dW_s - \int_t^T \int_E U_s^n(e)\tilde{\mu}(ds,de),\ 0 \leq t \leq T, \\ S_t \leq Y_t^n,\ 0 \leq t \leq T, \quad \text{and} \quad \int_0^T (Y_t^n - S_t)dK_t^n = 0. \end{cases} \tag{3}$$

respectively

$$\begin{cases} V_t = \xi + \int_t^T H(s,V_s,N_s,M_s)ds + \int_t^T g(s,V_s,N_s,M_s)d\overleftarrow{B}_s \\ + \int_t^T dK_s - \int_t^T N_s dW_s - \int_t^T \int_E M_s(e)\tilde{\mu}(ds,de),\ 0 \leq t \leq T, \\ S_t \leq V_t,\ 0 \leq t \leq T, \quad \text{and} \quad \int_0^T (V_t - S_t)dK_t = 0, \end{cases}$$

where $H(s,\omega,V,N,M) = C(1+|V|+|N|+|M|)$.

Lemma 4.2. (i) *a.s. for all, t and* $\forall n \leq m$, $Y_t^n \leq Y_t^m \leq V_t$,
(ii) *assume that* $(H.1)$, $(H.3)-(H.7)$ *is in force. Then there exists a constant* $A > 0$ *depending only on C,* α, ξ *and T such that:*

$$\|U^n\|_{\mathrm{L}^2(0,T,\tilde{\mu},\mathbb{R})} \leq A, \qquad \|Z^n\|_{M^2(0,T,\mathbb{R}^d)} \leq A.$$

*Proof.*The prove of the (i) follow from comparison theorem. It remains to prove (ii), by lemma 2.1, we have

$$\begin{aligned}
&\mathsf{E}|Y_t^n|^2+\mathsf{E}\int_t^T|Z_s^n|^2ds+\mathsf{E}\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\\
&\le \mathsf{E}|\xi|^2+2\mathsf{E}\int_t^T Y_s^n f_n(s,Y_s^n,Z_s^n,U_s^n)ds+2\mathsf{E}\int_t^T Y_s^n dK_s^n+\mathsf{E}\int_t^T\|g(s,Y_s^n,Z_s^n,U_s^n)\|^2ds.
\end{aligned}$$

By (i) in lemma 4.1, we have

$$\begin{aligned}
2\mathsf{E}\int_t^T Y_s^n f_n(s,Y_s^n,Z_s^n,U_s^n)ds &\le 2C\mathsf{E}\int_t^T Y_s^n(1+|Y_s^n|+|Z_s^n|+|U_s^n|)ds\\
&\le \mathsf{E}\int_t^T|Y_s^n|^2ds+TC^2+2C\mathsf{E}\int_t^T|Y_s^n|^2ds+\frac{C^2}{\gamma_1}\mathsf{E}\int_t^T|Y_s^n|^2ds\\
&\quad+\gamma_1\mathsf{E}\int_t^T|Z_s^n|^2ds+\frac{C^2}{\gamma_2}\mathsf{E}\int_t^T|Y_s^n|^2ds+\gamma_2\mathsf{E}\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds,\\
&\le\left(1+2C+\frac{C^2}{\gamma_1}+\frac{C^2}{\gamma_2}\right)\mathsf{E}\int_t^T|Y_s^n|^2ds+TC^2\\
&\quad+\gamma_1\mathsf{E}\int_t^T|Z_s^n|^2ds+\gamma_2\mathsf{E}\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds.
\end{aligned}$$

Also by the hypothesis associated with g, we get

$$\begin{aligned}
\|g(s,Y_s^n,Z_s^n,U_s^n)\|^2 &\le (1+\epsilon)\|g(s,Y_s^n,Z_s^n,U_s^n)-g(s,0,0,0)\|^2+\frac{1+\epsilon}{\epsilon}\|g(s,0,0,0)\|^2,\\
&\le (1+\epsilon)C|Y_s^n|^2+(1+\epsilon)\alpha\{|Z_s^n|^2+|U_s^n|^2\}+\frac{1+\epsilon}{\epsilon}\|g(s,0,0,0)\|^2.
\end{aligned}$$

Chossing $\gamma_1=\gamma_2=\frac{\epsilon^2}{2}$. Then, we obtain the following inequality

$$\begin{aligned}
&\mathsf{E}\left(|Y_t^n|^2+\int_t^T|Z_s^n|^2ds+\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\right)\\
&\le \mathsf{E}|\xi|^2+TC^2+\left(1+2C+\frac{4C^2}{\epsilon^2}+(1+\epsilon)C\right)\mathsf{E}\int_0^T|Y_s^n|^2ds+2\int_0^T Y_s^n dK_s^n\\
&+\left(\frac{\epsilon^2}{2}+(1+\epsilon)\alpha\right)\left\{\mathsf{E}\int_t^T|Z_s^n|^2ds+\mathsf{E}\int_0^T\int_E|U_s^n(e)|^2\lambda(de)ds\right\}+\frac{1+\epsilon}{\epsilon}\mathsf{E}\int_0^T\|g(s,0,0,0)\|^2ds.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\mathsf{E}\int_t^T\left(|Z_s^n|^2+\int_E|U_s^n(e)|^2\lambda(de)\right)ds\\
&\le\left(\frac{\epsilon^2}{2}+(1+\epsilon)\alpha\right)\mathsf{E}\left\{\int_t^T|Z_s^n|^2ds+\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\right\}+\Lambda+\theta\mathsf{E}|K_T^n-K_t^n|^2,
\end{aligned}$$

where

$$\Lambda=\mathsf{E}|\xi|^2+TC^2+\frac{1+\epsilon}{\epsilon}\mathsf{E}\int_t^T\|g(s,0,0,0)\|^2ds+\frac{1}{\theta}\mathsf{E}\left(\sup_{0\le S\le T}(S_s)^2\right)+T\left(1+2C+\frac{4C^2}{\epsilon^2}+(1+\epsilon)\right.$$
$$\left.\mathsf{E}\left(\sup_t|Y_t^n|^2\right)\right).$$

Now chossing ϵ and α such that $0\le\frac{\epsilon^2}{2}+(1+\epsilon)\alpha<1$, we obtain

$$\mathsf{E}\int_t^T|Z_s^n|^2ds+\mathsf{E}\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\leq\Lambda+\theta\mathsf{E}\left|K_T^n-K_t^n\right|^2. \tag{4}$$

On the other hand, we have from Eq.(3),

$$\begin{aligned}K_T^n-K_t^n &= Y_t^n-\xi-\int_t^T f_n(s,Y_s^n,Z_s^n,U_s^n)ds-\int_t^T g(s,Y_s^n,Z_s^n,U_s^n)d\overleftarrow{B}_s\\ &+\int_t^T Z_s^n dW_s+\int_t^T\int_E U_s^n(e)\tilde{\mu}(ds,de).\end{aligned}$$

Using the Hölder's inequality and assupmtion $(H.6)$, we have

$$\mathsf{E}|K_T^n-K_t^n|^2\leq C_1+C_2\Big(\mathsf{E}\int_t^T|Z_s^n|^2ds+\mathsf{E}\int_t^T\int_E|U_t^n|^2\lambda(de)ds\Big),$$

From inequality (4), we get

$$\mathsf{E}\int_0^T\Big(|Z_s^n|^2+\int_E|U_s^n(e)|^2\lambda(de)\Big)ds\leq\Lambda+\theta C_1+\theta C_2\mathsf{E}\int_t^T\Big(|Z_s^n|^2+\int_E|U_t^n|^2\lambda(de)\Big)ds.$$

Finally chossing θ such that $0\leq\theta C_2\leq 1$, we obtain

$$\mathsf{E}\int_t^T|Z_s^n|^2ds+\mathsf{E}\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\leq\Lambda+\theta C_1<\infty.$$

Thus the prove of this lemma is complet. ■

Lemma 4.3. *Assume that* $(H.1)$, $(H.3)-(H.7)$ *is in force. Then the sequence* (Z^n,U^n) *converges a.s. in* $M^2(0,T,\mathbb{R}^d)\times L^2(0,T,\tilde{\mu},\mathbb{R})$.

Proof. Let $n_0\geq C$. From Eq.(4.1), we deduce that there exists a process $Y\in\mathsf{S}^2(0,T,\mathbb{R})$ such that $Y^n\to Y$ a.s., as $n\to\infty$. Applying lemma 2.1 to $|Y_t^n-Y_t^m|^2$, for $n,m\geq n_0$

$$\begin{aligned}&\mathsf{E}\Big(|Y_t^n-Y_t^m|^2+\int_t^T|Z_s^n-Z_s^m|^2ds+\int_t^T\int_E|U_s^n(e)-U_s^m(e)|^2\lambda(de)ds\Big)\\ &\leq 2\mathsf{E}\int_t^T(Y_s^n-Y_s^m)(f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^m))ds\\ &+2\mathsf{E}\int_t^T(Y_s^n-Y_s^m)(dK_s^n-dK_s^m)+\mathsf{E}\int_t^T\|g(s,Y_s^n,Z_s^n,U_s^n)-g(s,Y_s^m,Z_s^m,U_s^m)\|^2ds.\end{aligned}$$

Since $\int_t^T(Y_s^n-Y_s^m)(dK_s^n-dK_s^m)\leq 0$, we deduce that

$$\begin{aligned}&\mathsf{E}\int_t^T|Z_t^n-Z_t^m|^2ds+\mathsf{E}\int_t^T\int_E|U_s^n(e)-U_s^m(e)|^2\lambda(de)ds\\ &\leq 2\mathsf{E}\int_t^T(Y_s^n-Y_s^m)(f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^m))ds\\ &+\mathsf{E}\int_t^T\|g(s,Y_s^n,Z_s^n,U_s^n)-g(s,Y_s^m,Z_s^m,U_s^m)\|^2ds.\end{aligned}$$

Using Hölder's inequality and assumption $(H.6)$ for g, we deduce that

$$\begin{aligned}&(1-\alpha)\mathsf{E}\Big\{\int_t^T|Z_t^n-Z_t^m|^2ds+\int_t^T\int_E|U_s^n(e)-U_s^m(e)|^2\lambda(de)ds\Big\}\\ &\leq 2\mathsf{E}\Big(\int_t^T|f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^m)|^2ds\Big)^{\frac{1}{2}}\mathsf{E}\Big(\int_t^T|Y_s^n-Y_s^m|^2ds\Big)^{\frac{1}{2}}\\ &+C\mathsf{E}\int_t^T|Y_s^n-Y_s^m|^2ds.\end{aligned}$$

Applying assumption $(H.6)$ for f and the boundedness of the sequence (Y^n,Z^n,U^n), we deduce that

$$(1-\alpha)\left\{\mathsf{E}\int_t^T|Z_t^n - Z_t^m|^2 ds + \mathsf{E}\int_t^T\int_E|U_s^n(e) - U_s^m(e)|^2\lambda(de)ds\right\} \leq C^{te}\mathsf{E}\int_t^T|Y_s^n - Y_s^m|^2 ds,$$

where the constant $C^{te} > 0$ depend only ξ, C, α and T.
Which yields that $(Z^n)_{n\geq 0}$ respectively $(U^n)_{n\geq 0}$ is a Cauchy sequence in $M^2(0,T,\mathbb{R}^d)$, respectively in $\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})$. Then there exists $(Z,U) \in M^2(0,T,\mathbb{R}^d)\times\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})$ such that

$$\mathsf{E}\int_0^T|Z_s^n - Z_s|^2 ds + \mathsf{E}\int_0^T\int_E|U_s^n(e) - U_s(e)|^2\lambda(de)ds \to 0, \quad \textit{as } n\to\infty.$$ ■

Theorem 4.1. *Assume that* $(H.1)$, $(H.3)-(H.7)$ *holds. Then Eq* (1) *admits a solution* $(Y,Z,U,K) \in D^2(\mathbb{R})$. *Moreover there is a minimal solution* (Y^*,Z^*,U^*) *of RBDSDEP* (1) *in the sense that for any other solution* (Y,Z,U) *of Eq.* (1), *we have* $Y^* \leq Y$.

Proof . From Eq.(4.1), it's readily seen that (Y^n) converges in $S^2(0,T,\mathbb{R})$, $dt\otimes dP - a.s.$ to $Y\in S^2(0,T,\mathbb{R})$. Otherwise thanks to lemma 4.3 there exists two subsequences still noted as the whole sequence $(Z^n)_{n\geq 0}$ respectively $(U^n)_{n\geq 0}$ such that

$$\mathsf{E}\int_0^T|Z_s^n - Z_s|^2 ds \to 0 \text{ as } n\to\infty, \quad \text{and} \quad \mathsf{E}\int_0^T\int_E|U_s^n(e) - U_s(e)|^2\lambda(de)ds \to 0, \text{ as } n\to\infty.$$

Applying lemma 4.1, we have $f_n(t,Y^n,Z^n,U^n) \to f(t,Y,Z,U)$ and the linear growth of f_n implies

$$|f_n(t,Y_t^n,Z_t^n,U_t^n)| \leq C\left(1+\sup_n\,(|Y_t^n| + |Z_t^n| + |U_t^n|)\right) \in \mathsf{L}^1([0,T];dt).$$

Thus by Lebesgue's dominated convergence theorem, we deduce that for almost all ω and uniformly in t, we have

$$\mathsf{E}\int_t^T f_n(s,Y_s^n,Z_s^n,U_s^n)ds \to \mathsf{E}\int_t^T f(s,Y_s,Z_s,U_s)ds.$$

We have by $(H.6)$ the following estimation

$$\mathsf{E}\int_t^T\|g(s,Y_s^n,Z_s^n,U_s^n) - g(s,Y_s,Z_s,U_s)\|^2 ds$$
$$\leq C\mathsf{E}\int_t^T|Y_s^n - Y_s|^2 ds + \alpha\mathsf{E}\int_t^T|Z_s^n - Z_s|^2 ds + \alpha\mathsf{E}\int_t^T\int_E|U_s^n(e) - U_s(e)|^2\lambda(de)ds \to 0,$$

as $n\to\infty$,

Using Burkholder-Davis-Gundy inequality, we have

$$\begin{cases} \mathbb{E} \sup_{0 \le t \le T} \left| \int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s \right|^2 \to 0, \\ \mathbb{E} \sup_{0 \le t \le T} \left| \int_t^T \int_E U_s^n(e) \tilde{\mu}(ds, de) - \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de) \right|^2 \to 0, \\ \mathbb{E} \sup_{0 \le t \le T} \left| \int_t^T g(s, Y_s^n, Z_s^n, U_s^n) d\overleftarrow{B}_s - \int_t^T g(s, Y_s, Z_s, U_s) d\overleftarrow{B}_s \right|^2 \to 0, \text{ in probability as, } n \to \infty. \end{cases}$$

Let the following reflected BDSDEPs with data (ξ, f, g, S)

$$\begin{cases} \hat{Y} \in S^2(0,T,\mathbb{R}), \quad \hat{Z} \in \mathrm{M}^2(0,T,\mathbb{R}^d), \quad K \in \mathrm{A}^2, \quad \hat{U} \in \mathrm{L}^2(0,T,\tilde{\mu},\mathbb{R}), \\ \hat{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + \int_t^T g(s, Y_s, Z_s, U_s) d\overleftarrow{B}_s + \int_t^T dK_s \\ -\int_t^T \hat{Z}_s dW_s - \int_t^T \int_E \hat{U}_s(e) \tilde{\mu}(ds, de), \\ S_t \le \hat{Y}_t, \quad 0 \le t \le T \quad \text{and} \quad \int_0^T \left(\hat{Y}_t - S_t \right) dK_t = 0. \end{cases}$$

By Itô's formula, we obtain

$$\begin{aligned} \mathbb{E}\left| Y_t^n - \hat{Y}_t \right|^2 &\le 2\mathbb{E} \int_t^T \left(Y_s^n - \hat{Y}_s \right) (f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)) ds \\ &+ 2\mathbb{E} \int_t^T \left(Y_s^n - \hat{Y}_s \right) (dK_s^n - dK_s) + \mathbb{E} \int_t^T \| g(s, Y_s^n, Z_s^n, U_s^n) - g(s, Y_s, Z_s, U_s) \|^2 ds \\ &- \mathbb{E} \int_t^T \int_E \left| U_s^n(e) - \hat{U}_s(e) \right|^2 \lambda(de) ds - \mathbb{E} \int_t^T \left| Z_s^n - \hat{Z}_s \right|^2 ds. \end{aligned}$$

Using the fact that $\mathbb{E} \int_t^T \left(Y_s^n - \hat{Y}_s \right) (dK_s^n - dK_s) \le 0,$ we get

$$\begin{aligned} &\mathbb{E}\left| Y_t^n - \hat{Y}_t \right|^2 + \mathbb{E} \int_t^T \int_E \left| U_s^n(e) - \hat{U}_s(e) \right|^2 \lambda(de) ds + \mathbb{E} \int_t^T \left| Z_s^n - \hat{Z}_s \right|^2 ds \\ &\le 2\mathbb{E} \int_t^T \left(Y_s^n - \hat{Y}_s \right) (f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)) ds \\ &+ \mathbb{E} \int_t^T \| g(s, Y_s^n, Z_s^n, U_s^n) - g(s, Y_s, Z_s, U_s) \|^2 ds. \end{aligned}$$

The by letting $n \to \infty$, we have $Y_t = \hat{Y}_t,\ U_t = \hat{U}_t$ and $Z_t = \hat{Z}_t\ dP \times dt - a.e.$
Let (Y^*, Z^*, U^*, K^*) be a solution of (1). Then by theorem 3.1, we have for any $n \in \mathbb{N}^*$, $Y^n \le Y^*$. Therefore, Y is a minimal solution of (1). ■

5. RBDSDEPs With Discontinuous Coefficients

In this section we are interested in weakening the conditions on *f*. We assume that *f* satisfy the following assumptions:

(**H.8**) There exists a nonnegative process $f_t \in \mathsf{M}^2(0,T,\mathbb{R})$ and constant $C > 0$, such that

$$\forall (t,y,z,u) \in [0,T] \times B^2(\mathbb{R}),\ |f(t,y,z,u)| \leq f_t(\omega) + C(|y| + |z| + |u|).$$

(**H.9**) $f(t,\cdot,z,u) : \mathbb{R} \to \mathbb{R}$ is a left continuous and $f(t,y,\cdot,\cdot)$ is a continuous.

(**H.10**) There exists a continuous fonction $\pi : [0,T] \times B^2(\mathbb{R})$ satisfying for $y \geq y'$, $(z,u) \in \mathbb{R}^d \times L^2(E,\mathsf{E},\lambda,\mathbb{R})$

$$\begin{cases} |\pi(t,y,z,u)| \leq C(|y| + |z| + |u|), \\ f(t,\omega,y,z,u) - f\left(t,\omega,y',z',u'\right) \geq \pi\left(t,y-y',z-z',u-u'\right). \end{cases}$$

(**H.11**) g satisfies $(H.2,(ii))$.

5.1. **Existence result**

The two next lemmas will be useful in the sequel.

Lemma 5.1. *Assume that π satisfies $(H.10)$, g satisfies $(H.11)$ and h belongs in $M^2(0,T,\mathbb{R})$. For a continuous processes of finite variation A belong in A^2 we consider the processes $(\bar{Y},\bar{Z},\bar{U}) \in S^2(0,T,\mathbb{R}) \times M^2(0,T,\mathbb{R}^d) \times L^2(0,T,\tilde{\mu},\mathbb{R})$ such that:*

$$\begin{cases} (i)\ \bar{Y}_t = \xi + \int_t^T (\pi(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s) + h(s))ds + \int_t^T g(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)d\overleftarrow{B}_s \\ \\ + \int_t^T dA_s - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_E \bar{U}_s(e)\tilde{\mu}(ds,de),\ 0 \leq t \leq T, \\ \\ (ii)\ \int_0^T \bar{Y}_t^- dA_s \geq 0. \end{cases} \tag{5}$$

Then we have,

(1) *The RBDSDEPs* (5.1) *admits a minimal solution* $(\bar{Y}_t,\bar{Z}_t,A_t,\bar{U}_t) \in D^2(\mathbb{R})$.

(2) *If* $h(t) \geq 0$ *and* $\xi \geq 0$, *we have* $\bar{Y}_t \geq 0,\ dP \times dt - a.s.$

Proof. (1) Has been obtained from a previous part.

(2) Applying lemma 2.1 to $|\bar{Y}_t^-|^2$, leads to

$$\begin{aligned} &\mathsf{E}\left(|\bar{Y}_t^-|^2 + \int_t^T 1_{\{\bar{Y}_s < 0\}} \|\bar{Z}_s\|^2 ds + \int_t^T \int_E 1_{\{\bar{Y}_s < 0\}} |\bar{U}_s(e)|^2 \lambda(de) ds\right) \\ &\quad \leq \mathsf{E}\Big(|\xi^-|^2 - 2\int_t^T \bar{Y}_s^-(\pi(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s) + h(s))ds - 2\int_t^T \bar{Y}_s^- dA_s \\ &\quad + \int_t^T 1_{\{\bar{Y}_s < 0\}} \|g(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)\|^2 ds\Big). \end{aligned}$$

Since $h(t) \geq 0,\ \xi \geq 0,$ and using the fact that $\int_0^T \bar{Y}_t^- dA_s \geq 0$, we obtain

$$\mathsf{E}|\bar{Y}_t^-|^2 + \mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\|\bar{Z}_s\|^2 ds + \mathsf{E}\int_t^T\int_E 1_{\{\bar{Y}_s<0\}}|\bar{U}_s(e)|^2\lambda(de)ds$$
$$\leq -2\mathsf{E}\int_t^T \bar{Y}_s^-\pi(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)ds + \mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\|g(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)\|^2 ds.$$

According to assumptions $(H.11)$, we have

$$\mathsf{E}|\bar{Y}_t^-|^2 + \mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\|\bar{Z}_s\|^2 ds + \mathsf{E}\int_t^T\int_E 1_{\{\bar{Y}_s<0\}}|\bar{U}_s(e)|^2\lambda(de)ds$$
$$\leq -2\mathsf{E}\int_t^T \bar{Y}_s^-\pi(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)ds + C\mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}|\bar{Y}_s^-|^2 ds$$
$$+\alpha\mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\|\bar{Z}_s\|^2 ds + \alpha\mathsf{E}\int_t^T\int_E 1_{\{\bar{Y}_s<0\}}|\bar{U}_s(e)|^2\lambda(de)ds.$$

Then by applying assumption $(H.10)$ and using Young's inequality, we can write

$$-2\mathsf{E}\int_t^T \bar{Y}_s^-\pi(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)ds \leq 2C\mathsf{E}\int_t^T|\bar{Y}_s^-|^2 ds + \frac{1}{2\epsilon}\mathsf{E}\int_t^T|\bar{Y}_s^-|^2 ds + 2\epsilon C^2\mathsf{E}\int_t^T\|\bar{Z}_s\|^2 ds$$
$$+\frac{1}{2\epsilon}\mathsf{E}\int_t^T|\bar{Y}_s^-|^2 ds + 2\epsilon C^2\mathsf{E}\int_t^T\int_E|\bar{U}_s(e)|^2\lambda(de)ds.$$

Then

$$\mathsf{E}|\bar{Y}_t^-|^2 + \mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\|\bar{Z}_s\|^2 ds + \mathsf{E}\int_t^T\int_E 1_{\{\bar{Y}_s<0\}}|\bar{U}_s(e)|^2\lambda(de)ds$$
$$\leq (3C+\epsilon^{-1})\mathsf{E}\int_t^T|\bar{Y}_s^-|^2 ds + (\alpha+2\epsilon C^2)\mathsf{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\left(\|\bar{Z}_s\|^2 + \int_E|\bar{U}_s(e)|^2\lambda(de)\right)ds.$$

Therefore, choosing ϵ, α and C such that $0<\alpha+2\epsilon C^2<1$ and using Gronwall's inequality, we have

$$\mathsf{E}|\bar{Y}_t^-|^2 = 0,$$

$\mathbf{P}-a.s.$ for all $t\in[0,T]$. This implies that $\bar{Y}_t\geq 0$, $\mathbf{P}-a.s.$ for all $t\in[0,T]$. ■

Now by theorem 4.1, we consider the processes $(\tilde{Y}_t^0,\tilde{Z}_t^0,\tilde{K}_t^0,\tilde{U}_t^0)$, $(Y_t^0,Z_t^0,K_t^0,U_t^0)$ and the sequence of processes $(\tilde{Y}_t^n,\tilde{Z}_t^n,\tilde{K}_t^n,\tilde{U}_t^n)_{n\geq 0}$ respectively as minimal solution, for all $t\in[0,T]$, to the following RBDSDEPs

$$\begin{cases} (i)\,\tilde{Y}_t^0 = \xi + \int_t^T\left[-C\left(\left|\tilde{Y}_S^0\right| + \left|\tilde{Z}_S^0\right| + \left|\tilde{U}_S^0\right|\right) - f_s\right]ds + \int_t^T g(s,\tilde{Y}_S^0,\tilde{Z}_S^0,\tilde{U}_S^0)d\overleftarrow{B}_S \\ +\int_t^T d\tilde{K}_S^0 - \int_t^T \tilde{Z}_S^0 dW_S - \int_t^T\int_E \tilde{U}_S^0(e)\tilde{\mu}(ds,de),\ 0\leq t\leq T, \\ (ii)\,\tilde{Y}_t^0 \geq S_t, \\ (iii)\int_0^T\left(\tilde{Y}_S^0 - S_S\right)d\tilde{K}_S^0 = 0, \end{cases} \qquad (6)$$

$$\begin{cases} (i)\ Y_t^0 = \xi + \int_t^T \Big[C\Big(\left| Y_s^0 \right| + \left| Z_s^0 \right| + \left| U_s^0 \right| \Big) + f_s \Big] ds + \int_t^T g(s, Y_s^0, Z_s^0, U_s^0) d\overleftarrow{B}_s \\ + \int_t^T dK_s^0 - \int_t^T Z_s^0 dW_s - \int_t^T \int_E U_s^0(e) \tilde{\mu}(ds, de),\ 0 \le t \le T, \\ (ii)\ Y_t^0 \ge S_t, \\ (iii)\ \int_0^T \Big(Y_s^0 - S_s \Big) dK_s^0 = 0, \end{cases} \tag{7}$$

and

$$\begin{cases} (i)\ \tilde{Y}_t^n = \xi + \int_t^T \Big[f(s, \tilde{Y}_s^{n-1}, \tilde{Z}_s^{n-1}, \tilde{U}_s^{n-1}) ds + \pi\Big(s, \tilde{Y}_t^n - \tilde{Y}_t^{n-1}, \tilde{Z}_t^n - \tilde{Z}_t^{n-1}, \tilde{U}_t^n - \tilde{U}_t^{n-1} \Big) \Big] ds \\ + \int_t^T g(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) d\overleftarrow{B}_s + \int_t^T d\tilde{K}_s^n - \int_t^T \tilde{Z}_s^n dW_s - \int_t^T \int_E \tilde{U}_s^n(e) \tilde{\mu}(ds, de),\ 0 \le t \le T, \\ (ii) \tilde{Y}_t^n \ge S_t, \\ (iii) \int_0^T (\tilde{Y}_s^n - S_s) d\tilde{K}_s^n = 0. \end{cases} \tag{8}$$

Lemma 5.2. *Under the assumptions* $(H.1)$, $(H.3)$, $(H.5)$ *and* $(H.8) - (H.11)$, *and for any* $n \ge 1$ *with* $t \in [0, T]$, *there holds*
$\tilde{Y}_t^0 \le \tilde{Y}_t^n \le \tilde{Y}_t^{n+1} \le Y_t^0$.

Proof. For any $n \ge 0$, we set

$$\begin{cases} \delta\rho_t^{n+1} = \rho_t^{n+1} - \rho_t^n, \\ \text{and} \\ \Delta\psi^{n+1}(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}) = \psi(s, \delta\tilde{Y}_s^{n+1} + \tilde{Y}_s^n, \delta\tilde{Z}_s^{n+1} + \tilde{Z}_s^n, \delta\tilde{U}_s^{n+1} + \tilde{U}_s^n) - \psi(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n). \end{cases}$$

Invoke (8) to write

$$\delta\tilde{Y}_t^{n+1} = \int_t^T [\pi(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}) + \theta_s^n] ds + \int_t^T \Delta g^{n+1}(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}) d\overleftarrow{B}_s$$
$$+ \int_t^T d(\delta\tilde{K}_s^{n+1}) - \int_t^T \delta\tilde{Z}_s^{n+1} dW_s - \int_t^T \int_E \delta\tilde{U}_s^{n+1}(e) \tilde{\mu}(ds, de),$$

where

$$\begin{cases} \theta_s^n = \Delta f^n(s, \delta\tilde{Y}_s^n, \delta\tilde{Z}_s^n, \delta\tilde{U}_s^n) - \pi(s, \delta\tilde{Y}_s^n, \delta\tilde{Z}_s^n, \delta\tilde{U}_s^n), \\ and \\ \theta_s^0 = f(s, \tilde{Y}_s^0, \tilde{Z}_s^0, \tilde{U}_s^0) + C(|\tilde{Y}_s^0| + |\tilde{Z}_s^0| + |\tilde{U}_s^0|) + f_s, \forall n \geq 0. \end{cases}$$

According to its definition, one can show that θ_s^0 and Δg^n, $\forall n \geq 0$ satisfy all assumption of lemma 5.1. Moreover, since $\tilde{K}_t^n$ is a continuous and increasing process, for all $n \geq 0$, then $\delta\tilde{K}_s^{n+1}$ is a continuous process of finite variation. Using the same arguments as in first part, it is possible to show that

$$\int_0^T (\tilde{Y}_t^{n+1} - \tilde{Y}_t^n)^- d(\delta\tilde{K}_t^{n+1}) = \int_0^T (\tilde{Y}_t^{n+1} - \tilde{Y}_t^n)^- d\tilde{K}_t^{n+1} - \int_0^T (\tilde{Y}_t^{n+1} - \tilde{Y}_t^n)^- d\tilde{K}_t^n \geq 0.$$

By applying lemma 5.1, we deduce that $\delta\tilde{Y}_t^{n+1} \geq 0$, i.e. $\tilde{Y}_t^{n+1} \geq \tilde{Y}_t^n$, for all $t \in [0,T]$. So we have $\tilde{Y}_t^{n+1} \geq \tilde{Y}_t^n \geq \tilde{Y}_t^0$.

Now we shall prove that $\tilde{Y}_t^{n+1} \leq Y_t^0$. By definition, we have

$$\begin{aligned} Y_t^0 - \tilde{Y}_t^{n+1} &= \int_t^T (-C(|Y_s^0 - \tilde{Y}_s^{n+1}| + |Z_s^0 - \tilde{Z}_s^{n+1}| + |U_s^0 - \tilde{U}_s^{n+1}|) + \Lambda_s^{n+1}) ds \\ &+ \int_t^T (g(s, Y_s^0, Z_s^0, U_s^0) - g(s, \tilde{Y}_s^{n+1}, \tilde{Z}_s^{n+1}, \tilde{U}_s^{n+1})) d\overleftarrow{B}_s \\ &+ \int_t^T (dK_s^0 - d\tilde{K}_s^{n+1}) + \int_t^T (Z_s^0 - \tilde{Z}_s^{n+1}) dW_s - \int_t^T \int_E (U_s^0(e) - \tilde{U}_s^{n+1}(e)) \tilde{\mu}(ds, de), \end{aligned}$$

where

$$\begin{aligned} \Lambda_s^{n+1} = {} & C(|Y_s^0 - \tilde{Y}_s^{n+1}| + |Z_s^0 - \tilde{Z}_s^{n+1}| + |U_s^0 - \tilde{U}_s^{n+1}| + |Y_s^0| + |Z_s^0| + |U_s^0|) + f_s - f(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) \\ & - \pi(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}). \end{aligned}$$

Also by repeated use of lemma 5.1, we deduce that $Y_t^0 - \tilde{Y}_t^{n+1} \geq 0$, i.e. $Y_t^0 \geq \tilde{Y}_t^{n+1}$, for all $t \in [0,T]$. Thus, we have for all $n \geq 0$, $Y_t^0 \geq \tilde{Y}_t^{n+1} \geq \tilde{Y}_t^n \geq \tilde{Y}_t^0$, $dP \times dt - a.s.$, $\forall t \in [0,T]$. ■

Now we can state our main result.

Theorem 5.1. *Under assumption* $(H.1)$, $(H.3)$, $(H.5)$ *and* $(H.8) - (H.11)$, *the RBDSDEPs* (1) *has a minimal solution* $(Y_t, Z_t, K_t, U_t)_{t \in [0,T]}$ $(D^2(\mathbb{R})$.

Proof. Since $|\tilde{Y}_t^n| \leq \max(\tilde{Y}_t^0, Y_t^0) \leq |\tilde{Y}_t^0| + |Y_t^0|$ for all $t \in [0,T]$, we have

$$\sup_n \mathsf{E}\left(\sup_{0 \leq t \leq T} |\tilde{Y}_t^n|^2\right) \leq \mathsf{E}\left(\sup_{0 \leq t \leq T} |\tilde{Y}_t^0|^2\right) + \mathsf{E}\left(\sup_{0 \leq t \leq T} |Y_t^0|^2\right) < \infty.$$

Therefore, we deduce from the Lebesgue's dominated convergence theorem that $(\tilde{Y}_t^n)_{n \geq 0}$ converges in $S^2(0,T,\mathbb{R})$ to a limit Y.

On the other hand, by (8) we can write

$$\tilde{Y}_0^{n+1} = \tilde{Y}_T^{n+1} + \int_0^T [f(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)ds + \pi(s,\delta\tilde{Y}_t^{n+1},\delta\tilde{Z}_t^{n+1},\delta\tilde{U}_t^{n+1})]ds$$
$$+\int_0^T g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})d\overleftarrow{B}_s + \int_0^T d\tilde{K}_s^{n+1} - \int_0^T \tilde{Z}_s^{n+1}dW_s - \int_0^T\int_E \tilde{U}_s^{n+1}(e)\tilde{\mu}(ds,de).$$

Apply then lemma 2.1 to obtain

$$\mathsf{E}\left|\tilde{Y}_0^{n+1}\right|^2 + \mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)ds$$
$$\leq \mathsf{E}|\xi|^2 + 2\mathsf{E}\int_0^T \tilde{Y}_s^{n+1}(f(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) + \pi(s,\delta\tilde{Y}_s^{n+1},\delta\tilde{Z}_s^{n+1},\delta\tilde{U}_s^{n+1}))ds$$
$$+2\mathsf{E}\int_0^T \tilde{Y}_s^{n+1}d\tilde{K}_s^{n+1} + \mathsf{E}\int_0^T\left|\left|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})\right|\right|^2 ds.$$

By $(H.8)$ and $(H.10)$, it follows that

$$\tilde{Y}_t^{n+1}(f(t,\tilde{Y}_t^n,\tilde{Z}_t^n,\tilde{U}_t^n) + \pi(t,\delta\tilde{Y}_t^{n+1},\delta\tilde{Z}_t^{n+1},\delta\tilde{U}_t^{n+1}))$$
$$\leq \left|\tilde{Y}_t^{n+1}\right|\{f_t(\omega) + 2C(\left|\tilde{Y}_t^n\right| + \left|\tilde{Z}_t^n\right| + \left|\tilde{U}_t^n\right|) + C(\left|\tilde{Y}_t^{n+1}\right| + \left|\tilde{Z}_t^{n+1}\right| + \left|\tilde{U}_t^{n+1}\right|)\}$$
$$\leq \frac{\left|\tilde{Y}_t^{n+1}\right|^2}{2} + \frac{f_t(\omega)}{2} + C^2\left|\tilde{Y}_t^{n+1}\right|^2 + \left|\tilde{Y}_t^n\right|^2 + \frac{2C^2}{\epsilon_1}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_1}{2}\left|\tilde{Z}_t^n\right|^2 + \frac{2C^2}{\epsilon_2}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_2}{2}\left|\tilde{U}_t^n\right|^2$$
$$+ C\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{C^2}{2\epsilon_3}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_3}{2}\left|\tilde{Z}_t^{n+1}\right|^2 + \frac{C^2}{2\epsilon_4}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_4}{2}\left|\tilde{U}_t^{n+1}\right|^2$$
$$= \left(\frac{1}{2} + C^2 + \frac{2C^2}{\epsilon_1} + \frac{2C^2}{\epsilon_2} + \frac{C^2}{2\epsilon_3} + \frac{C^2}{2\epsilon_4} + C\right)\left|\tilde{Y}_t^{n+1}\right|^2$$
$$+ \frac{\epsilon_3}{2}\left|\tilde{Z}_t^{n+1}\right|^2 + \frac{\epsilon_4}{2}\left|\tilde{U}_t^{n+1}\right|^2 + \left|\tilde{Y}_t^n\right|^2 + \frac{\epsilon_1}{2}\left|\tilde{Z}_t^n\right|^2 + \frac{\epsilon_2}{2}\left|\tilde{U}_t^n\right|^2 + \frac{f_t(\omega)}{2}.$$

Also applying $(H.11)$ leads to the following inequality

$$\left|\left|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})\right|\right|^2 \leq \left|\left|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1}) - g(s,0,0,0)\right|\right|^2 + \|g(s,0,0,0)\|^2$$
$$\leq C\left|\tilde{Y}_s^{n+1}\right|^2 + \alpha\left\{\left|\tilde{Z}_s^{n+1}\right|^2 + \left|\tilde{U}_s^{n+1}\right|^2\right\} + \|g(s,0,0,0)\|^2.$$

Then using Young's inequality, allows writing

$$2\mathsf{E}\int_0^T \tilde{Y}_s^{n+1}d\tilde{K}_s^{n+1} \leq 2\mathsf{E}\int_0^T S_s d\tilde{K}_s^{n+1} \leq \tfrac{1}{\theta}\mathsf{E}\left(\sup_{0\leq t\leq T}|S_t|^2\right) + \theta\mathsf{E}\left|\tilde{K}_T^{n+1}\right|^2.$$

Therefore, there exists a constant C independent of n such that for any ϵ_i, where $i = 1:4$, we have

$$\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)ds$$
$$\leq C + (\epsilon_3+\alpha)\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + (\epsilon_4+\alpha)\mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}\right|^2\lambda(de)ds \tag{9}$$
$$+\epsilon_1\mathsf{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds + \epsilon_2\mathsf{E}\int_t^T\int_E\left|\tilde{U}_s^n\right|^2\lambda(de)ds + \theta\mathsf{E}\left|\tilde{K}_T^{n+1}\right|^2.$$

Moreover, since

$$\tilde{K}_T^{n+1} = \tilde{Y}_0^{n+1} - \xi - \int_0^T [f(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)ds + \pi(s,\delta\tilde{Y}_s^{n+1},\delta\tilde{Z}_s^{n+1},\delta\tilde{U}_s^{n+1})]ds$$
$$-\int_0^T g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})d\overleftarrow{B}_s + \int_0^T \tilde{Z}_s^{n+1}dW_s + \int_0^T\int_E \tilde{U}_s^{n+1}(e)\tilde{\mu}(ds,de),$$

we may use Hölder's inequality and assumptions $(H.8)$ and $(H.10)$. Consequently, there exists two constants C_1 and C_2 depending on $\xi, C, \alpha, \epsilon_i, i = 1,\ldots,4$, and we have that

$$\mathsf{E}\left|\tilde{K}_T^{n+1}\right|^2 \le C_1 + C_2\left(\mathsf{E}\int_0^T\left(\left|\tilde{Z}_s^n\right|^2 + \left|\tilde{Z}_s^{n+1}\right|^2\right)ds + \mathsf{E}\int_0^T\int_E\left(\left|\tilde{U}_s^n\right|^2 + \left|\tilde{U}_s^{n+1}\right|^2\right)\lambda(de)ds\right).$$

We return back to inequality (9) and write

$$\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)ds$$
$$\le (C+\theta C_1) + (\epsilon_1+\theta C_2)\mathsf{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds + (\epsilon_2+\theta C_2)\mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^n\right|^2\lambda(de)ds$$
$$+(\epsilon_3+\alpha+\theta C_2)\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + (\epsilon_4+\alpha+\theta C_2)\mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}\right|^2\lambda(de)ds.$$

Then by taking $\epsilon_1 = \epsilon_2 = \epsilon_0$ and $\epsilon_3 = \epsilon_4 = \bar{\epsilon}$, we have

$$\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)ds$$
$$\le (C+\theta C_1) + (\epsilon_0+\theta C_2)\left\{\mathsf{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^n\right|^2\lambda(de)ds\right\}$$
$$+(\bar{\epsilon}+\theta C_2+\alpha)\mathsf{E}\int_0^T\left(\left|\tilde{Z}_s^{n+1}\right|^2 + \int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\right)ds.$$

A further choise of $\bar{\epsilon}$, θ and α such that $0 \le (\bar{\epsilon}+\theta C_2+\alpha) < 1$, allows for

$$\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)ds$$
$$\le (C+\theta C_1) + (\epsilon_0+\theta C_2)\left\{\mathsf{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^n\right|^2\lambda(de)ds\right\}$$
$$\le (C+\theta C_1)\sum_{i=0}^{i=n-1}(\epsilon_0+\theta C_2)^i + (\epsilon_0+\theta C_2)^n\left\{\mathsf{E}\int_0^T\left|\tilde{Z}_s^0\right|^2 ds + \mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^0\right|^2\lambda(de)ds\right\}.$$

Now choosing ϵ_0, θ and C_2 such that $\epsilon_0+\theta C_2 < 1$, and noting that $\mathsf{E}\int_0^T\left(\left|\tilde{Z}_s^0\right|^2 + \int_E\left|\tilde{U}_s^0\right|^2\lambda(de)\right)ds < \infty$, allows for

$$\sup_{n\in\mathbb{N}}\mathsf{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds < \infty \quad\text{and}\quad \sup_{n\in\mathbb{N}}\mathsf{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)ds < \infty.$$

Consequently, we deduce that

$$\mathsf{E}\left|\tilde{K}_T^{n+1}\right|^2 < \infty.$$

Now we shall prove that $(\tilde{Z}^n,\tilde{K}^n,\tilde{U}^n)$ is a Cauchy sequence in $\mathsf{M}^2(0,T,\mathbb{R}^d)\times A^2\times\mathsf{L}^2(0,T,\tilde{\mu},\mathbb{R})$, set $\Gamma_s^n = f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1},\tilde{U}_s^{n-1}) + \pi(s,\delta\tilde{Y}_s^n,\delta\tilde{Z}_s^n,\delta\tilde{U}_s^n)$ to arrive at

$$\tilde{Y}_t^n - \tilde{Y}_t^m = \int_t^T (\Gamma_s^n - \Gamma_s^m)ds + \int_t^T (g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m))d\overleftarrow{B}_s$$
$$+ \int_t^T (d\tilde{K}_s^n - d\tilde{K}_s^m) - \int_t^T (\tilde{Z}_s^n - \tilde{Z}_s^m)dW_s - \int_t^T \int_E (\tilde{U}_s^n(e) - \tilde{U}_s^m(e))\tilde{\mu}(ds,de).$$

Further application of lemma 2.1 to $\left|\delta \tilde{Y}_s^{n,m}\right|^2 = \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2$, results with

$$\mathsf{E}\left|\tilde{Y}_t^n - \tilde{Y}_t^m\right|^2 + \mathsf{E}\int_t^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathsf{E}\int_t^T \int_E \left|\tilde{U}_s^n - \tilde{U}_s^m\right|^2 \lambda(de)ds$$
$$\leq 2\mathsf{E}\int_t^T (\tilde{Y}_s^n - \tilde{Y}_s^m)(\Gamma_s^n - \Gamma_s^m)ds + 2\mathsf{E}\int_t^T (\tilde{Y}_s^{n+1} - \tilde{Y}_s^n)(d\tilde{K}_s^n - d\tilde{K}_s^m)$$
$$+ \mathsf{E}\int_t^T \left|\left|(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m))\right|\right|^2 ds.$$

Since $\int_t^T (\tilde{Y}_s^{n+1} - \tilde{Y}_s^n)(d\tilde{K}_s^n - d\tilde{K}_s^m) \leq 0,$ we obtain

$$\mathsf{E}\int_0^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathsf{E}\int_t^T \int_E \left|\tilde{U}_s^n - \tilde{U}_s^m\right|^2 \lambda(de)ds$$
$$\leq 2\mathsf{E}\int_t^T (\tilde{Y}_s^n - \tilde{Y}_s^m)(\Gamma_s^n - \Gamma_s^m)ds + \mathsf{E}\int_t^T \left|\left|(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m))\right|\right|^2 ds.$$

Apply next Hölder's inequality and assumption $(H.11)$ to write

$$(1-\alpha)\left\{\mathsf{E}\int_t^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathsf{E}\int_t^T \int_E \left|\tilde{U}_s^n - \tilde{U}_s^m\right|^2 \lambda(de)ds\right\}$$
$$\leq 2\mathsf{E}\left(\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds\right)^{\frac{1}{2}} \mathsf{E}\left(\int_t^T |\Gamma_s^n - \Gamma_s^m|^2 ds\right)^{\frac{1}{2}} + C\mathsf{E}\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds.$$

The boundedness of the sequence $(\tilde{Y}^n,\tilde{Z}^n,\tilde{K}^n,\tilde{U}^n)$ implies that $\Lambda = \sup_{n\in\mathbb{N}} \left(\mathsf{E}\int_0^T |\Gamma_s^n|^2 ds\right) < \infty,$ to allow for

$$(1-\alpha)\mathsf{E}\int_t^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathsf{E}\int_t^T \int_E \left|\tilde{U}_s^n - \tilde{U}_s^m\right|^2 \lambda(de)ds$$
$$\leq 4\Lambda\mathsf{E}\left(\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds\right)^{\frac{1}{2}} + C\mathsf{E}\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds,$$

which implies that each of $(\tilde{Z}^n)_{n\geq 0}$ and $(\tilde{U}^n)_{n\geq 0}$, is a Cauchy sequence in $M^2(0,T,\mathbb{R}^d)$ and in $L^2(0,T,\tilde{\mu},\mathbb{R})$ respectively. Then there exists $(Z,U) \in M^2(0,T,\mathbb{R}^d) \times L^2(0,T,\tilde{\mu},\mathbb{R})$ such that,

$$\mathsf{E}\int_t^T \left|\tilde{Z}_s^n - Z_s\right|^2 ds + \mathsf{E}\int_t^T \int_E \left|\tilde{U}_s^n - U_s\right|^2 \lambda(de) \to 0, \quad \text{as } n \to \infty. \tag{10}$$

On the other hand, applying the Burkholder-Davis-Gundy inequality and (10), ends up with

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T \tilde{Z}_s^n dW_s - \int_t^T Z_s dW_s\right|^2 \le \mathbb{E}\int_t^T \left|\tilde{Z}_s^n - Z_s\right|^2 ds \to 0, \text{ as } n\to\infty,$$

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T\int_E \tilde{U}_s^n(e)\tilde{\mu}(ds,de) - \int_t^T\int_E U_s(e)\tilde{\mu}(ds,de)\right|^2$$
$$\le \mathbb{E}\int_t^T\int_E\left|\tilde{U}_s^n - U_s\right|^2\lambda(de)ds \to 0, \text{ as } n\to\infty,$$

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)d\overleftarrow{B}_s - \int_t^T g(s,Y_s,Z_s,U_s)d\overleftarrow{B}_s\right|^2$$
$$\le C\,\mathbb{E}\int_t^T\left|\tilde{Y}_s^n - Y_s\right|^2 ds + \alpha\mathbb{E}\int_t^T\left|\tilde{Z}_s^n - Z_s\right|^2 ds + \alpha\mathbb{E}\int_t^T\int_E\left|\tilde{U}_s^n - U_s\right|^2\lambda(de)ds \to 0, \text{ as } n\to\infty.$$

Therefore, from the proprieties of (f,π), we have
$$\Gamma_s^n = f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1},\tilde{U}_s^{n-1}) + \pi(s,\delta\tilde{Y}_s^n,\delta\tilde{Z}_s^n,\delta\tilde{U}_s^n) \to f(s,Y_s,Z_s,U_s),$$
$P-a.s.$, for all $t\in[0,T]$ as $n\to\infty$. It follows then, by the dominated convergence theorem, that
$$\mathbb{E}\int_0^T|\Gamma_s^n - f(s,Y_s,Z_s,U_s)|^2 ds \to 0.$$
Since $(\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n,\Gamma_s^n)$ converges in $B^2(\mathbb{R})\times M^2(0,T,\mathbb{R})$ and

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{K}_t^n - \tilde{K}_t^m\right|^2\right) \le \mathbb{E}\left|\tilde{Y}_0^n - \tilde{Y}_0^m\right|^2 + \mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{Y}_t^n - \tilde{Y}_t^m\right|^2\right) + \mathbb{E}\int_0^T|\Gamma_s^n - \Gamma_s^m|^2 ds$$
$$+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t (g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m))d\overleftarrow{B}\right|^2$$
$$+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t(\tilde{Z}_s^n - \tilde{Z}_s^m)dW_s\right|^2 + \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_E(\tilde{U}_s^n(e) - \tilde{U}_s^m(e))\tilde{\mu}(ds,de)\right|^2,$$

for any $n,m\ge 0$, we deduce that
$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{K}_t^n - \tilde{K}_t^m\right|^2\right)\to 0,$$
as $n,m\to\infty$. Consequently, there exists a $\mathcal{F}_t$ –measurable process K wich value in $\mathbb{R}$ such that
$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{K}_t^n - K_t\right|^2\right)\to 0, \quad as\ n\to\infty. \tag{11}$$
Finally, we have

$$\mathsf{E}\left(\sup_{0\le t\le T}\left|\tilde{Y}_t^n - Y_t\right|^2 + \int_t^T\left|\tilde{Z}_s^n - Z_s\right|^2 ds + \int_t^T\int_E\left|\tilde{U}_s^n - U_s\right|^2\lambda(de)ds + \sup_{0\le t\le T}\left|\tilde{K}_t^n - K_t\right|^2\right) \to 0,$$
$$as\ n \to \infty.$$

Obviously, $K_0 = 0$ and $\{K_t; 0 \le t \le T\}$ is a increasing and continuous process. From (8), we have for all $n \ge 0$, $\tilde{Y}_t^n \ge S_t$, $\forall t \in [0,T]$.Then $Y_t \ge S_t$, $\forall t \in [0,T]$.

On the other hand, from the result of Saisho [12], it follows that

$$\int_0^T(\tilde{Y}_s^n - S_s)d\tilde{K}_s^n \to \int_0^T(Y_s - S_s)dK_s,\ \mathbf{P}-a.s.,\quad as\ n \to \infty.$$

Using the identity $\int_0^T(\tilde{Y}_s^n - S_s)d\tilde{K}_s^n = 0$ for all $n \ge 0$, we conclude that $\int_0^T(Y_s - S_s)dK_s = 0$. Letting then $n \to +\infty$ in equation (1), proves that $(Y_t, Z_t, K_t, U_t)_{t\in[0,T]}$ is a solution to *it*. Assume ultimately (Y_*, Z_*, U_*, K_*) to be a solution to (1) to invoke theorem 3.1, and observe that for any $n \in \mathbb{N}^*$, $Y^n \le Y_*$. Therefore, Y is a minimal. ■

Remark 5.1. Using the same arguments and the following approximating sequence

$$f_n(t,\omega,y,z,u) = \sup_{(y',z',u')\in B^2(\mathbb{R})}\left[f\left(t,\omega,y',z',u'\right) - n\left(\left|y-y'\right| + |z-z'| + \left|u-u'\right|\right)\right],$$

one can prove that the RBDSDE (1) has a maximal solution.

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