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Numerical Methods for Certain Classes of Markovian Backward Stochastic Differential Equations and Quasi-Linear Parabolic Partial Differential Equations Via Girsanov's Theorem

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Abstract. *In this paper, we apply Girsanov's theorem on the change of probability measures to obtain an approximation result and a numerical method for certain classes of Markovian coupled forward—backward stochastic differential equations. As a deterministic application, by using the Feynman-Kac's formula, we obtain a new time-space discretization scheme for certain classes of quasi-linear parabolic partial differential equations.*

Key words : Backward Stochastic Differential Equations, Parabolic Partial Differential Equations, Discrete Time Approximation, Monte Carlo Simulation, Girsanov's Theorem, Feynman-Kac's Formula.

AMS Subject Classifications : 60H10, 60K30, 60H35

1. Introduction

Backward stochastic differential equations, (BSDEs for short), represent a new class of stochastic differential equations, (SDEs), whose value is prescribed at a terminal time T. BSDEs have received considerable attention in the probability literature as they provide a probabilistic formula for the solution of certain classes of quasi-linear parabolic partial differential equations, (PDEs), that are related to viscosity solutions of these PDEs. The theory of BSDEs has found wide applications in areas such as stochastic control, theoretical economics and mathematical finance problems, (we refer the interested reader,e.g. , to El

Karoui et *al*. [5] for several applications to option pricing). As of 1973, linear BSDEs were first introduced by Bismut [2], who used them to study stochastic optimal control problems in the stochastic version of Pontryagin's maximum principle. Then in 1990, Pardoux and Peng [9] considered the general case of BSDEs and established existence and uniqueness of their solutions under some assumptions such as Lipschitzianity of the generator. More precisely, they studied BSDEs of the form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

where $(W_t)_{0\leq t\leq T}$ is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$, T is a fixed finite horizon, $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural Brownian filtration. The random function $f : [0, T] \times \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ is the generator of the BSDE, and the $\mathbb{R}^k$-valued $\mathcal{F}_T$-adapted random variable ξ is the terminal condition. The existence of the uniquely adapted solutions of these BSDEs is proved in [9] under the assumptions $(\aleph_1)$:

- f is uniformly Lipschitzian in (y, z), i.e., there exists a finite positive constant C_f, such that for all (y, z, y', z'),

$$|f(t, y, z) - f(t, y', z')| \leq C_f\,(|y - y'| + \|z - z'\|).$$

- ξ and $(f(t, 0, 0))_{t\in[0,T]}$ satisfies the square integrability condition:

$$\mathsf{E}\left(|\xi|^2 + \int_0^T |f(t, 0, 0)|^2\, dt\right) < +\infty.$$

It should be noticed that in the case of linear BSDEs, valued in $\mathbb{R}$, and defined as:

$$Y_t = \xi + \int_t^T (a_s Y_s + b_s^\top Z_s + c_s)ds - \int_t^T Z_s^\top dW_s, \quad 0 \leq t \leq T,$$

the previous assumptions become $(\aleph_2)$:

- a and b are bounded progressively measurable processes valued in $\mathbb{R}$ and $\mathbb{R}^k$,
- ξ and c satisfy:

$$\mathsf{E}\left(|\xi|^2 + \int_0^T |c_t|^2\, dt\right) < +\infty,$$

and we have explicitly

$$Y_t = \Gamma_t^{-1} \mathsf{E}\left[\xi \Gamma_T + \int_t^T c_s \Gamma_s ds \;\middle|\; \mathsf{F}_t\right], \tag{1}$$

where

$$\Gamma_t = \exp\left[\int_0^t b_s dW_s - \tfrac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t a_s ds\right].$$

Formula (1) will be useful to prove the main approximation in this paper. It was also used in [7] and [11] to prove a useful comparison theorem for BSDEs. Recently, in 1992, Pardoux and Peng [10] showed that the solution of some Markovian BSDEs is related to some forward classical SDEs, corresponds to a probabilistic solution of a non-linear PDEs, and obtained a generalization of the classical Feynman Kac's formula. This probabilistic representation leads to a numerical method for solving PDEs, relying on Monte-Carlo simulations of the forward diffusion process X, whose convergence rate does not depend on the dimension of the problem.

Let us consider the parabolic heat equation:

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2} \triangle_X v = 0, & [0,T) \times \mathbb{R}^k, \\ v(T,.) = h(.) & \mathbb{R}^k. \end{cases} \tag{2}$$

It is well known that the solution to (2) is given by

$$v(t,x) = \int h(y) \frac{1}{(4\pi(T-t))^{\frac{k}{2}}} e^{-\frac{(x-y)^2}{4(T-t)}} dy.$$

From the Gaussian distribution of W_t, we observe that the solution v can also be represented as:

$$v(t,x) = \mathsf{E}[h(x + W_{T-t})], \quad (t,x) \in [0,T] \times \mathbb{R}^k,$$

which gives a Monte-Carlo method for computing an approximation of v by the empirical mean:

$$v(t,x) \simeq \bar{v}^n(t,x) := \frac{1}{n} \sum_{i=1}^{n} h(x + W^i_{T-t}),$$

where $(W^i)_{1\le i \le n}$ is an n-sample drawn from an exact simulation of W. The convergence of $\bar{v}^n$ to v is ensured by the law of large numbers, when n goes to infinity. While the rate of convergence, obtained from the central limit theorem, is equal to $\frac{1}{\sqrt{n}}$, and independent of the dimension k of the heat equation. More generally, let us consider the linear PDE:

$$\begin{cases} \frac{\partial v}{\partial t} + \Im v + f = 0 & [0,T) \times \mathbb{R}^k, \\ v(T,.) = h(.) & \mathbb{R}^k, \end{cases} \tag{3}$$

where $\Im$ is the second order Dynkin operator:

$$\Im v = b(x).D_x v + \frac{1}{2} tr(\sigma(x)\sigma^\top(x) D_x^2 v).$$

Under standard conditions on the functions b, σ, f and h defined on $\mathbb{R}^k$, there exists a unique solution v to (3), which may be represented by the Feynman-Kac formula:

$$v(t,x) = \mathsf{E}\Big[\int_t^T f(s, X_s^{t,x}) ds + h(X_T^{t,x}) \Big], \quad (t,x) \in [0,T] \times \mathbb{R}^k, \tag{4}$$

where $X^{t,x}$ is the solution to the (forward) diffusion process:

$$X_s^{t,x} = x + \int_t^s b(X_u^{t,x}) du + \int_t^s \sigma(X_u^{t,x}) dW_u, \quad t \le s \le T,$$

starting from $x \in \mathsf{R}$ at time t. Notice that the Feynman-Kac formula (4) can easily be derived from Itô's formula when v is smooth. Indeed, in this case, by defining the pair of processes (Y,Z):

$$Y_t := v(t, X_t), \qquad Z_t := \sigma^\top(X_t) D_x. v(t, X_t), \quad 0 \le t \le T$$

and applying Itô's formula to $v(s, X_s)$ between t and T with v satisfying the PDE (3), we obtain

$$Y_t = h(X_T) + \int_t^T f(s, X_s) ds - \int_t^T Z_s dW_s, \ 0 \le t \le T. \tag{5}$$

Equation (5) can be viewed as a BSDE in the pair of adapted processes (Y,Z) with terminal condition $h(X_T)$. Then by taking the conditional expectation in (5), we retrieve the

Feynman-Kac's formula (4).

In this paper, we use the Girsanov's theorem on the change of probability measure to obtain an approximation result and a numerical method for certain classes of Markovian BSDEs. As a deterministic application, by using the Feynman-Kac's formula, we devise a time-space discretization scheme for certain classes of quasi-linear parabolic PDEs.

2. The Main Results

Here we entertain classes of one-dimensional Markovian BSDEs driven by one-dimensional Brownian motion.

2.1. **An approximation result for Markovian BSDEs**

Consider the Markovian BSDEs

$$Y_t = h(X_T) + \int_t^T f(s, X_S, Y_S, Z_S)ds - \int_t^T Z_S dW_S, \quad t \in [0, T], \tag{6}$$

where X is a forward diffusion process of dynamics

$$X_t = x + \int_0^t b(s,\ X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

with $x \in \mathbb{R}$ and b, σ, f and h are valued in $\mathbb{R}$ with Lipschitzianity and square integrability conditions. We invoke the following numerical approximation of the BSDE (6), and refer the interested reader, e.g., to [1], [3] and [6]. Let π be a partition of time points $0 = t_0 < t_1 <\ldots< t_n = T$ of $[0, T]$, with a fixed time step $\Delta t_i := t_{i+1} - t_i$ and a corresponding $\Delta W_{t_i} := W_{t_{i+1}} - W_{t_i}$ to write

$$\begin{cases} X^\pi_{t_0} = x,\ Y^\pi_{t_n} = h(X^\pi_T), \\ X^\pi_{t_{i+1}} = X^\pi_{t_i} + b(t_i, X^\pi_{t_i})\Delta t_i + \sigma(t_i, X^\pi_{t_i})\Delta W_{t_i}, \qquad i \le n, \\ Z^\pi_{t_i} = \mathsf{E}\left[Y^\pi_{t_i} \dfrac{\triangle W t_i}{\triangle t_i} \middle| \mathcal{F}_{t_i} \right], \qquad i \le n, \\ Y^\pi_{t_i} = \mathsf{E}[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}] + f(t_i, X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}) \triangle t_i, \qquad i \le n. \end{cases} \tag{7}$$

The practical implementation of the numerical scheme (7) requires the computation of conditional expectations which can be approximated for example by non-parametric regression methods, based on the projection on a set of basis functions, (see for example [6]), however this method is highly technical in practice which is the reason why BSDEs have not been used by practitioners yet.

Now we begin the statement and proof of the main result of this section. We start with nominal reference deterministic (non random) trajectories denoted by $(\bar{x}, \bar{y}, \bar{z})$, and defined viz

$$\begin{cases} \bar{x}_t = x + \int_0^t b(s,\bar{x}_s)ds, \\ \bar{z}_t = \mathsf{E}(Z_t), \\ \bar{y}_t = \mathsf{E}(h(X_T)) + \int_t^T \mathsf{E}(f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s))ds, \quad t \in [0,T]. \end{cases} \tag{8}$$

Clearly by the Lipschitzianity assumptions, the triplet $(\bar{x},\bar{y},\bar{z})$ exists. So it is possible to write

$$\begin{cases} \tilde{a}_s = \frac{\partial f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)}{\partial x}, \\ \tilde{b}_s = \frac{\partial f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)}{\partial y}, \\ \tilde{c}_s = \frac{\partial f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)}{\partial z}, \\ \tilde{e}_s = f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s) - \mathsf{E}(f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)) - \bar{z}_s\tilde{c}_s, \\ \widetilde{X}_s = X_s - \bar{x}_s, \\ \tilde{g}_s = \tilde{a}_s\widetilde{X}_s + \tilde{e}_s. \end{cases}$$

Theorem 2.1. *If the processes $\tilde{b}$ and $\tilde{g}$ satisfy the assumptions (H_2), and the process $\tilde{c}_s$ satisfies Novikov's condition, then there exists a new probability measure Q such that the following approximation*

$$Y^t_{estim} = \bar{y}_t + \Gamma_t^{-1}\mathsf{E}^Q\left[\Gamma_T(h(X_T) - \mathsf{E}(h(X_T))) + \int_t^T \tilde{g}_s\Gamma_s ds|\mathcal{F}_t\right],$$

of the solution Y to the BSDE (6), holds. Here "estim" stands for the estimated value.

Proof. We define the error as the difference between Y_t and $\bar{y}_t$ viz $\widetilde{Y}_t = Y_t - \bar{y}_t$. Next by combining (6) and (8), we obtain the following dynamics for $\widetilde{Y}_t$

$$\widetilde{Y}_t = \tilde{h}(X_T) + \int_t^T \tilde{f}(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{9}$$

where

$$\begin{cases} \tilde{h}(X_T) = h(X_T) - \mathsf{E}(h(X_T)), \\ \tilde{f}(s,X_s,Y_s,Z_s) = f(s,X_s,Y_s,Z_s) - \mathsf{E}(f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)). \end{cases}$$

To arrive at a linear approximation of the BSDE (9), we make a Taylor series expansion of $\tilde{f}(s,X_s,Y_s,Z_s)$ around $(\bar{x}_s,\bar{y}_s,\bar{z}_s)$, (assuming that the partial derivatives exist):

$$\tilde{f}(s,X_s,Y_s,Z_s) \approx \tilde{f}(s,\bar{x}_s,\bar{y}_s,\bar{z}_s) + \frac{\partial f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)}{\partial x}\widetilde{X}_s + \frac{\partial f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)}{\partial y}\widetilde{Y}_s + \frac{\partial f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s)}{\partial z}(Z_s - \bar{z}_s).$$

This leads to the approximate linear model:

$$\widetilde{Y}_t = \tilde{h}(X_T) + \int_t^T \left(\tilde{b}_s\widetilde{Y}_s + \tilde{c}_s Z_s + \tilde{g}_s\right)ds - \int_t^T Z_s dW_s, \quad t \in [0,T]. \tag{10}$$

Now, define

$$\widetilde{Z}_t = \exp\left\{-\int_0^t \tilde{c}_s dW_s - \frac{1}{2}\int_0^t \tilde{c}_s^2 ds\right\},$$
$$Q(A) = \int_A \widetilde{Z}_T\, dP, \qquad \forall A \in \mathcal{F},$$
$$W_t^Q = W_t - \int_0^t \tilde{c}_s\, ds,$$

and suppose that the process $\tilde{c}$ satisfies the Novikov's condition:

$$\mathsf{E}\left[\exp\left(\frac{1}{2}\int_0^T |\tilde{c}_s|^2 ds\right)\right] < \infty.$$

Then by the well known Girsanov's theorem (see for example cite: O), $(W_t^Q)_{0\le t\le T}$ is a Brownian motion under the new probability measure Q. Therefore, (10) becomes

$$\widetilde{Y}_t = \tilde{h}(X_T) + \int_t^T \left(\tilde{b}_S \widetilde{Y}_S + \tilde{g}_S\right) ds - \int_t^T Z_S dW_S^Q, \quad t \in [0,T]. \tag{11}$$

Satisfaction of the assumptions $(\aleph_2)$ by $\tilde{b}$ and $\tilde{g}$ and of (1), leads to

$$\begin{cases} \Gamma_t = \exp\left(\int_0^t \tilde{b}_s ds\right), \\ \widetilde{Y}_t = \Gamma_t^{-1} \mathsf{E}^Q\left(\Gamma_T \tilde{h}(X_T) + \int_t^T \tilde{g}_s \Gamma_s ds | \mathcal{F}_t\right), \end{cases}$$

which completes the proof. ■

Remark 2.1. If the generator f is deterministic (non random), then $\tilde{f}(s, \bar{x}_s, \bar{y}_s, \bar{z}_s) = 0$ and we have:

$$Y_{estim}^t = \bar{y}_t + \Gamma_t^{-1}\Gamma_T[\mathsf{E}^Q(h(X_T)|\mathcal{F}_t) - \mathsf{E}(h(X_T))] + \Gamma_t^{-1}\int_t^T \mathsf{E}^Q(\tilde{g}_s|\mathcal{F}_t)\Gamma_s ds.$$

Example 2.1. Consider the following nonlinear BSDE:

$$-dY_t = -Y_t(1 - Y_t)\left(\frac{3}{4} - Y_t\right)dt - Z_t dW_t, \quad \xi = h(X_T), \quad t \in [0,1], \tag{12}$$

where $h(y) = \frac{1}{1+\frac{y}{x}}$ and the process $X^t = x\exp\left(-W_t - \frac{t}{4}\right)$ is the solution to the forward diffusion process:

$$X_t = x + \int_0^t \frac{X_s}{4} ds - \int_0^t X_s\, dW_s, \quad x \in \mathbb{R}^*.$$

In this equation the process Y represents the potential of a membrane. This equation is called the stochastic FitzHugh-Nagumo equation, (see [12]) and is used in physics, genetics and biology, among other fields. The exact adapted solution of (12) is

$$(Y_{\text{real}}^t, Z_{\text{real}}^t) = \left(\frac{1}{1+\exp\left(-W_t - \frac{t}{4}\right)}, \frac{\exp\left(-W_t - \frac{t}{4}\right)}{\left(1+\exp\left(-W_t - \frac{t}{4}\right)\right)^2}\right).$$

The relationship:

$$-Y_s(1-Y_s)(\tfrac{3}{4}-Y_s) = \tfrac{Y_sZ_s}{4}(1-\tfrac{3}{x}X_s),$$

indicates that the BSDE (12) is equal to the BSDE:

$$Y_t = h(X_T) + \int_t^T \frac{Y_sZ_s}{4} 1 - \frac{3}{x}X_s ds - \int_t^T Z_s\, dW_s. \tag{13}$$

Clearly the generator in (12) satisfies the Lipschitzianity assumptions, but the generator in (13) does not. However, we can use (13) to test the sensitivity of our numerical method to dependencies of the driver of the BSDE on the processes X and Z, (see the numerical results obtained below).

In this case, we have: $\bar{x}_t = x\exp\left(\frac{t}{4}\right)$ and

$$\begin{cases} f(s,\bar{x}_s,\bar{y}_s,\bar{z}_s) = \frac{\bar{y}_s\bar{z}_s}{4}(1-\frac{3}{x}\bar{x}_s), \quad \tilde{f}(s,\bar{x}_s,\bar{y}_s,\bar{z}_s) = 0, \\ \tilde{a}_s = \frac{-3}{4x}\bar{y}_s\bar{z}_s, \quad \tilde{b}_s = \frac{\bar{z}_s}{4}(1-\frac{3}{x}\bar{x}_s), \\ \tilde{c}_s = \frac{\bar{y}_s}{4}(1-\frac{3}{x}\bar{x}_s), \quad \tilde{e}_s = \frac{\bar{y}_s\bar{z}_s}{4}(1-\frac{3}{x}\bar{x}_s). \end{cases}$$

Therefore

$$\begin{aligned} \tilde{Y}_t &= \exp\left(\int_t^T \frac{\bar{z}_s}{4}(1-\tfrac{3}{x}\bar{x}_s)ds\right)[\mathsf{E}^Q(h(X_T))|\mathcal{F}_t - \mathsf{E}(h(X_T))] \\ &+ \exp\left(-\int_0^t \frac{\bar{z}_s}{4}(1-\tfrac{3}{x}\bar{x}_s)ds\right)\int_t^T \frac{-3}{4x}\bar{y}_s\bar{z}_s\left(-2\bar{x}_s+\tfrac{x}{3}\right)\exp\left(\int_0^s \frac{\bar{z}_u}{4}(1-\tfrac{3}{x}\bar{x}_u)du\right)ds \\ &+ \exp\left(-\int_0^t \frac{\bar{z}_s}{4}(1-\tfrac{3}{x}\bar{x}_s)ds\right)\int_t^T \frac{-3}{4}\bar{y}_s\bar{z}_s[\mathsf{E}^Q(\exp(-W_s-\tfrac{s}{4}))|\mathcal{F}_t] \\ &\quad \exp\left[\int_0^s \frac{\bar{z}_u}{4}(1-\tfrac{3}{x}\bar{x}_u)du\right]ds \\ &:= I_t^1 + I_t^2 + \exp\left(-W_t-\tfrac{t}{2}\right)I_t^3, \end{aligned}$$

where we have used in the last equality the fact that by the martingale property, we have :

$$\mathsf{E}^Q[\exp(-W_s-\tfrac{s}{4})|\mathcal{F}_t] = \exp\left(-W_t-\tfrac{t}{2}\right)\exp(\tfrac{s}{4}).$$

Finally, we will develop a discretization time scheme and a Monte Carlo simulation (with $\triangle t_i = \frac{T}{n}$) as follows :

1. $Y_{estim}^T = Y_T = \xi$.
2. The deterministic nominal reference trajectory $\bar{z}$ is given by (7) as:

   $$\bar{z}_{t_i} = \frac{n}{T}\mathsf{E}\left[Y_{estim}^{t_{i+1}} \triangle W_{t_i}\right],$$

   then, we apply the Monte Carlo simulation.
3. The deterministic nominal reference trajectory $\bar{y}$ is given by the Euler scheme:

   $$\begin{cases} \bar{y}_T = \mathsf{E}(h(X_T)), \\ \bar{y}_{t_i} = \bar{y}_{t_{i+1}} + \int_{t_i}^{t_{i+1}} \frac{\bar{y}_s\bar{z}_s}{4}\left(1-\frac{3}{x}\bar{x}_s\right)ds, \quad i \le n. \end{cases}$$

Therefore

$$\begin{cases} \bar{y}_T = \mathsf{E}(h(X_T)), \\ \bar{y}_{t_i} \approx \dfrac{\bar{y}_{t_{i+1}}}{1-\frac{T}{4n}\bar{z}_{t_i}+\frac{3T}{4xn}\bar{x}_{t_i}\bar{z}_{t_i}}, \quad i \le n. \end{cases}$$

4. We consider $I_t^1 = \exp(I_t)\left[\mathsf{E}^Q(h(X_T)|\mathcal{F}_t) - \mathsf{E}(h(X_T))\right]$, where I_t is obtained by the Euler scheme:

$$\begin{cases} I_T = 0, \\ I_{t_i} \approx I_{t_{i+1}} + \dfrac{\bar{z}_{t_i}}{4}\left(1 - \dfrac{3}{x}\bar{x}_{t_i}\right)\dfrac{T}{n}, \quad i \le n. \end{cases}$$

Use, further, the Markov property and the Monte Carlo method for simulation of the term $\left[\mathsf{E}^Q(h(X_T)|\mathcal{F}_t) - \mathsf{E}^Q(h(X_T))\right]$.

5. Then let

$$J_t^2 = \frac{I_t^2}{\exp\left(-\int_0^t \frac{\bar{z}_s}{4}(1-\frac{3}{x}\bar{x}_s)ds\right)} = \int_t^T \frac{-3}{4x}\bar{y}_s\bar{z}_s\left(-2\bar{x}_s + \frac{x}{3}\right)$$
$$\left[\exp\int_0^s \frac{\bar{z}_u}{4}\left(1 - \frac{3}{x}\bar{x}_u\right)du\right]ds,$$

to write

$$\begin{cases} J_T^2 = 0, \\ J_{t_i}^2 = J_{t_i+1}^2 - \dfrac{3}{4x}\bar{y}_{t_i}\bar{z}_{t_i} - \left(2\bar{x}_{t_i} + \dfrac{x}{3}\right)\exp\left[\displaystyle\int_0^{t_i}\frac{\bar{z}_s}{4}\left(1 - \frac{3\bar{x}_s}{x}\right)ds\right]\dfrac{T}{n}, \end{cases}$$

and

$$\frac{I_{t_i}^2}{\exp\left[-\int_0^{t_i}\frac{\bar{z}_s}{4}(1-\frac{3}{x}\bar{x}_s)ds\right]} = \frac{I_{t_i+1}^2}{\exp\left[-\int_0^{t_i+1}\frac{\bar{z}_s}{4}(1-\frac{3}{x}\bar{x}_s)ds\right]}$$
$$- \frac{3}{4x}\bar{y}_{t_i}\bar{z}_{t_i}\left(-2\bar{x}_{t_i} + \frac{x}{3}\right)\exp\left[\int_0^{t_i}\frac{\bar{z}_s}{4}\left(1 - \frac{3}{x}\bar{x}_s\right)ds\right]\frac{T}{n}.$$

Therefore

$$\begin{cases} I_T^2 = 0, \\ I_{t_i}^2 \approx I_{t_{i+1}}^2 \exp(I_{t_i} - I_{t_{i+1}}) - \dfrac{3}{4x}\bar{y}_{t_i}\bar{z}_{t_i}\left(-2\bar{x}_{t_i} + \dfrac{x}{3}\right)\dfrac{T}{n}, \quad i \le n. \end{cases}$$

6. In the same way, it is possible to write

$$\begin{cases} I_T^3 = 0, \\ I_{t_i}^3 \approx I_{t_{i+1}}^3 \exp(I_{t_i} - I_{t_{i+1}}) - \dfrac{3}{4}\bar{y}_{t_i}\bar{z}_{t_i}\left[\exp\left(\dfrac{t_i}{4}\right)\right]\dfrac{T}{n}, \quad i \le n. \end{cases}$$

7. Finally, we have : $Y^t_{estim} = \widetilde{Y}_t + \bar{y}_t$.

By using our main (R)-codes, we obtain for $x = 1$ the results exhibited in Figures 1 and 2.

Figure 1 : Trajectories of Yreal and Yestim via (12)

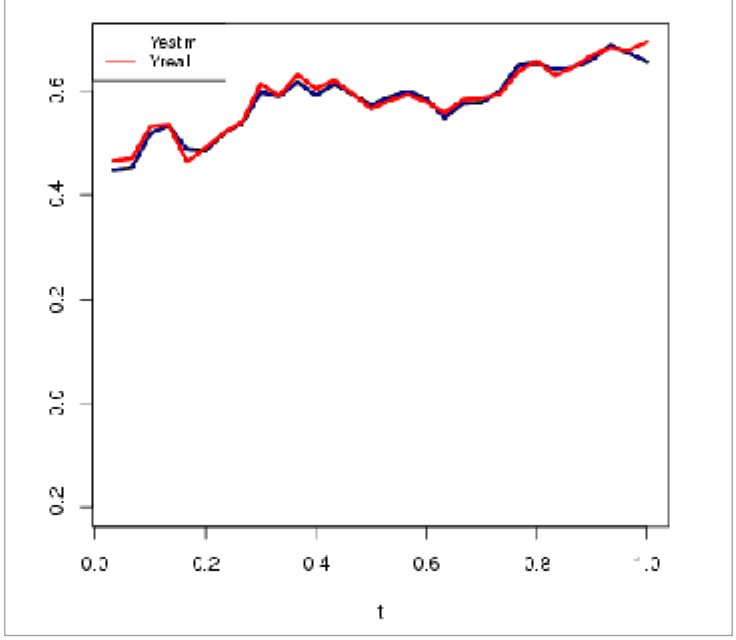
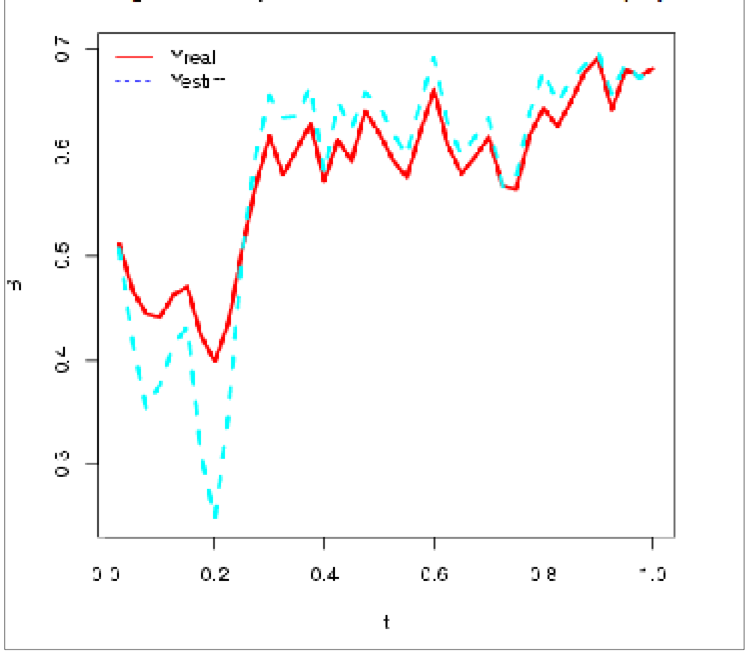


Figure 2 : Trajectories of Yreal and Yestim via (13)



Clearly, we can see the sensitivity of our numerical method to linkage of the driver of the BSDE with the processes X and Z.

2.2. **Markovian BSDEs and PDEs**

In this subsection, as an application of theorem 2.1 we use a generalization of the classical Feynman-Kac's formula, which establishes a connection between BSDEs and PDEs. For more details of this fact, we refer the interested reader, to El Karoui et al. [5] and [10]. This leads to a time-space discretization scheme for certain classes of quasi-linear PDEs.

For any given $(t,x) \in [0,T] \times \mathbb{R}$, consider the following classical Itô's SDE, defined on $[0,T]$,

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u, \quad t \leq s \leq T, \tag{14}$$

starting from $x \in \mathbb{R}$ at time t. We then consider the associated BSDE,

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(s, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})du - \int_s^T Z_u^{t,x} dW_u, \quad t \leq s \leq T. \tag{15}$$

Here, standard Lipschitzianity conditions are assumed on the coefficients.

The measurability properties of the solution $X_s^{t,x}, s \in [t,T]$ of (14) still hold for the solution $(Y_s^{t,x}, Z_s^{t,x}), s \in [t,T]$ of (15). More precisely, the solution of the BSDE (15) is adapted to the future σ-algebra of W after t, that is, it is $\mathcal{F}_s^t$-adapted where for each $s \in [t,T]$, $\mathsf{F}_s^t = \sigma(W_u - W_t,\ t \leq u \leq s)$, (see Proposition 4.2, in [5], page 44). Let v be a function that is smooth enough to be able to apply Itô's formula to $v(s, X_s^{t,x})$. u is supposed to be the solution of the following quasi-linear PDE:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \Im v + f\left(t, x, v(t,x), \sigma(t,x)\dfrac{\partial v}{\partial x}\right) = 0 & [0,T) \times \mathbb{R}, \\ v(T,.) = h(.) & \mathbb{R}, \end{cases} \tag{16}$$

where $\Im$ is the second order Dynkin operator:

$$\Im v = b(t,x)\frac{\partial v}{\partial x} + \tfrac{1}{2}\sigma^2(t,x)\frac{\partial^2 v}{\partial x^2}.$$

Then $v(t,x) = Y_t^{t,x}$ which is deterministic, with $(Y_s^{t,x}, Z_s^{t,x}), s \in [t,T]$ is the unique solution of BSDE (15). Also, we have:

$$(Y_s^{t,x}, Z_s^{t,x}) = v(s, X_s^{t,x}), \sigma(s, X_s^{t,x})\frac{\partial v}{\partial x}(s, X_s^{t,x}),\ t \leq s \leq T. \tag{17}$$

Example 2.2. Consider the so-called “deterministic KPZ' equation, (see [4]):

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}(t,x) + \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}(t,x)\right)^2 = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ v(T,x) = h(x), & x \in \mathbb{R}, \end{cases} \tag{18}$$

Such an equation admits too a "Cole-Hopf explicit solution" that writes as

$$v(t,x) = \log \mathsf{E}[\exp(h(x + W_{T-t}))].$$

Clearly the BSDE associated with (18) is:

$$Y_t = h(X_T) + \int_t^T \frac{1}{2} Z_s^2 ds - \int_t^T Z_s dW_s, \tag{19}$$

with $b = 0$, $\sigma = 1$ and $X_t = x + W_t$.

The changes of variables $P_t = \exp(Y_t)$ and $Q_t = Z_t \exp(Y_t)$ with the Itô's formula, lead to the equation:

$$P_t = \exp(h(X_T)) - \int_t^T Q_s dW_t,$$

and the solution of (19) is given by:

$$Y_t = \ln \mathsf{E}(\exp(h(X_T))|\mathcal{F}_t).$$

Using the Markov property of W, we have:

$$Y_t = \ln \mathsf{E}(\exp(h(Y + x + W_t))),$$

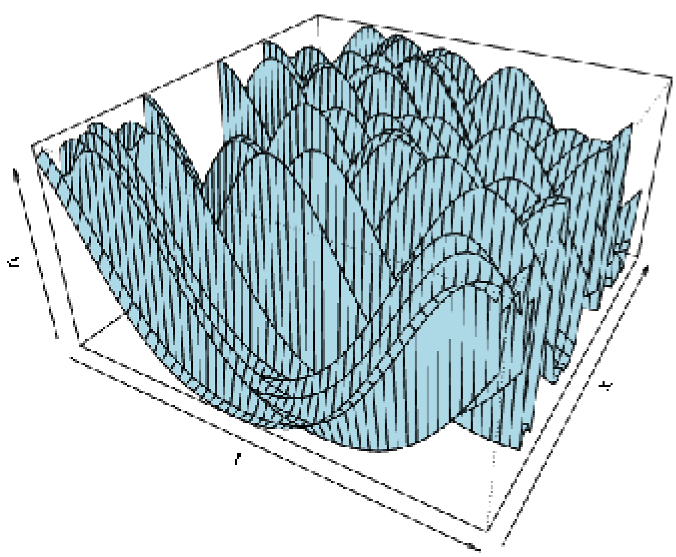
where $Y = W_{T-t} \sim \mathsf{N}(0, T-t)$.

Therefore

$$v(t,x) = Y_0 = \log \mathsf{E}[\exp(h(x + W_{T-t}))] \quad .$$

Finally, by using the Monte Carlo simulation for $h(x) = \sin(2\pi x)$ and our (R)-code, we obtain the results for $v(t,x)$, shown in Figure 3.

Figure 3 : Monte Carlo method for *v(t,x)*



Consider now the translated Brownian motion B and its associated filtration defined by: $B_s = W_{s+t} - W_t$, $\mathcal{F}'_s = \mathcal{F}^t_{s+t}$, $0 \le s \le T-t$. Let X'_s, $0 \le s \le T-t$ be the $\mathcal{F}'_s$ adapted solution of the SDE

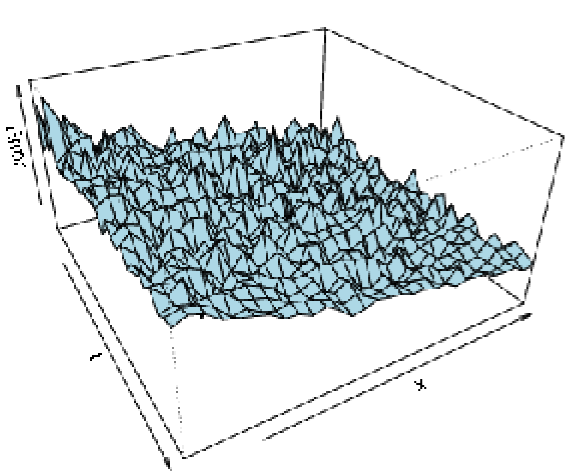
$$X'_s = x + \int_0^s b(u, X'_u)du + \int_0^s \sigma(u, X'_u)dB_u,$$

and let $(Y'_s, Z'_s,\ 0 \le s \le T-t)$ be the solution of the associated BSDE. By uniqueness, we have $X^{t,x}_s = X'_{s-t}$ and $(Y'_{s-t}, Z'_{s-t}) = (Y^{t,x}_s, Z^{t,x}_s)$, $t \le s \le T$. Consequently, $X^{t,x}_s$ and $(Y^{t,x}_s, Z^{t,x}_s)$ are $\mathcal{F}^t_s$

adapted. Finally, we are able to simulate the solution of the PDE (16) as follows:

1. We apply theorem 2.1 to (X', Y', Z'),
2. Use the formulas: $Y_s^{t,x} = Y'_{s-t}$ and $v(t,x) = Y_t^{t,x}$,
3. Finally, take $v(t,x) = Y'_0$.

Example 2.3. Use the same example given in the previous application of theorem 2.1, (subsection 2.1). Here we have the estimated solution: $v(t,x) = Y^0_{estim}$ and the real (exact) solution given by (17). Computations of the error in $v(t,x)_{estim}$ relative to $v(t,x)$ are finally exhibited in Figure 4.



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