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Controllability Results for Time-Dependent Impulsive Neutral Stochastic Functional Differential Equations Driven by Fractional Brownian Motion

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Abstract. *In this paper we consider the controllability of a certain class of non-autonomous impulsive neutral evolution stochastic functional differential equations, with time varying delays, driven by a fractional Brownian motion, in a Hilbert space. Sufficient conditions for this controllability are obtained by employing a fixed point approach. A practical example is provided to illustrate the viability of the abstract result of this work.*

Key words : Controllability, Neutral Stochastic Function Differential Equations, Evolution Operator, Fractional Brownian Motion.

AMS Subject Classifications : 35R10, 60H15, 60G15, 60J65

1. Introduction

Controllability is a qualitative property of dynamical control systems and is of particular importance in control theory. It is one of the fundamental concept in mathematical control theory and plays an important role in both deterministic and stochastic control theories. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals. If the system cannot be controlled completely, then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may be referred to papers [4, 5, 10, 26] and references therein.

In addition, impulsive effects exist widely in many evolution processes in which states are changed instantaneously at certain moments, involving fields such as finance, economics, neural networks, electronics and telecommunications. There has been a significant development in the study of impulsive stochastic partial differential equations as in [2, 22, 23, 27]. In many areas of science, there has been an increasing interest in the investigation of systems incorporating memory or after effect, i.e. when there is the effect of delay on state equations. Many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems. Quite recently, neutral stochastic functional differential equations driven by fractional Brownian motion have attracted the interest of many researchers as in [7, 11, 12, 13, 14, 15, 16, 17] and the references therein.

The literature concerning the existence and qualitative properties of solutions of time-dependent functional stochastic differential equations is very restricted and limited to a very few articles. We mention in this respect the recent paper by Ren et al. [24] concerning the existence of mild solutions for a class of stochastic evolution equations driven by fractional Brownian motion in Hilbert space. This fact is the main motivation of our work.

Motivated by the above works, this paper is concerned with the controllability of time-dependent impulsive neutral functional stochastic differential equations described by

$$\begin{cases} d[x(t)+g(t,x(t-r(t)))] = [A(t)x(t)+f(t,x(t-\rho(t))) + Bu(t)]dt+\sigma(t)dB^H(t), \\ t\in[0,T],\, t\neq t_k \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k\, x(t_k^-), \quad t=t_k,\ k=1,2,\ldots,m, \\ x(t)=\varphi(t), \ -\tau\leq t\leq 0 \quad a.s. \quad \tau>0, \end{cases} \tag{1}$$

in a real Hilbert space X with inner product $<.,.>$ and norm $\|.\|$, where $(A(t), t\in[0,T])$ is a family of linear closed operators from a space X into X that generates an evolution system of operators $\{U(t,s), 0\leq s\leq t\leq T\}$. B^H is a fractional Brownian motion on a real and separable Hilbert space Y. Moreover, the fixed moments of time t_k satisfy $0<t_1<t_2<\ldots<t_m$, $\Delta x(t_k)$ denotes the jump in the state x at time t_k with $I(.) : X\longrightarrow X$ determining the size of the jump, r, $\rho:[0,+\infty)\rightarrow[0,\tau]$ $(\tau>0)$ are continuous and $f,g:[0,+\infty)\times X\rightarrow X$, $\sigma:[0,+\infty)\rightarrow \mathcal{L}_2^0(Y,X)$, are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

To the best of our knowledge, there are no papers which investigate the study of controllability for time-dependent impulsive neutral stochastic functional differential equations with delays driven by fractional Brownian motion. Thus, we will make the first attempt to study such problem in this paper.

Our results are inspired by the work in [24] where the existence and uniqueness of mild solutions to model (1) with $B=0$, and $\Delta x(t_k)=0$ is studied, as well as some results on the asymptotic behavior.

The rest of this paper is organized as follows. Section 2, recapitulates some notations, basic concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and recalls some preliminary results about the evolution operator. Section 3, gives sufficient conditions to prove the controllability for the problem (1. In Section 4 we give an example to illustrate the efficiency of the obtained result.

2. Preliminaries

2.1. **Evolution families**

In this subsection we introduce the notion of an evolution family.

Definition 2.1. A set $\{U(t,s) : 0 \le s \le t \le T\}$ of bounded linear operators on a Hilbert space X is called an *evolution family* if
(a) $U(t,s)U(s,r) = U(t,r)$, $U(s,s) = I$ if $r \le s \le t$,
(b) $(t,s) \to U(t,s)x$ is strongly continuous for $t > s$.

Let $\{A(t), t \in [0,T]\}$ be a family of closed densely defined linear unbounded operators on the Hilbert space X, with domain $D(A(t))$ independent of t, satisfying the following conditions introduced in [1].

There exist constants $\lambda_0 \ge 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L, K \ge 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{K}{1+|\lambda|}, \tag{2}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)R(\lambda_0, A(t)) \ - R(\lambda_0, A(s))\| \le L|t-s|^\mu |\lambda|^{-\nu}. \tag{3}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta$ where $\Sigma_\theta := \lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \le \theta$.

It is well known, that this assumption implies that there exists a unique evolution family $\{U(t,s) : 0 \le s \le t \le T\}$ on X such that $(t,s) \to U(t,s) \in \mathsf{L}(X)$ is continuous for $t > s$, $U(\cdot, s) \in \mathsf{C}^1((s,\infty), \mathsf{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \le C\,(t-s)^{-k}, \tag{4}$$

for $0 < t - s \le 1$, $k = 0, 1$, $0 \le \alpha < \mu$, $x \in D((\lambda_0 - A(s))^\alpha)$, and a constant C depending only on the constants in (2)-(3). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ generates $\{U(t,s) : 0 \le s \le t \le T\}$. Note that $U(t,s)$ is exponentially bounded by (4) with $k = 0$.

Remark 2.1. If $\{A(t), t \in [0,T]\}$ is a second order differential operator A, that is $A(t) = A$ for each $t \in [0,T]$, then A generates a C_0 –semigroup $\{e^{At}, t \in [0,T]\}$.

For additional details on evolution systems and their properties, we refer the reader to [29].

2.2. **Fractional Brownian motion**

For the convenience for the reader, we revisit briefly here some of the basic results of fractional Brownian motion calculus. For more details, we refer the reader to [21] and the references therein.

Let X and Y be two real, separable Hilbert spaces and let $\mathsf{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathsf{L}(Y,X)$. Let $Q \in \mathsf{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \ge 0$ $(n = 1, 2\ldots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2 \ldots)$ is a complete orthonormal basis in Y. Let $B^H = (B^H(t))$ be $Y-$ valued fbm on $(\Omega, \mathsf{F}, \mathsf{P})$ with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. This process,

$$\mathsf{E}\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \quad x, y \in Y \quad t, s \in [0,T],$$

is Gaussian as it starts from 0, has zero mean and covariance. In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathsf{L}_2^0 := \mathsf{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathsf{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathsf{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\, \psi e_n\|^2 < \infty,$$

and that the space L_2^0 equipped with the inner product $\langle \varphi, \psi\rangle_{\mathsf{L}_2^0} = \sum_{n=1}^{\infty}\langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\phi(s); s \in [0,T]$ be a function with values in $\mathsf{L}_2^0(Y,X)$, such that

$$\sum_{n=1}^{\infty} \|K^* \phi Q^{\frac{1}{2}} e_n\|_{\mathsf{L}_2^0}^2 < \infty.$$

The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda n}\, \phi(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda n}\, (K_H^*(\phi e_n)(s) d\beta_n(s), \tag{5}$$

where β_n is the standard Brownian motion used to present β_n^H as in the Wiener integral representation.

Now, we end this subsection by stating the following lemma which is fundamental for proving our main result.

Lemma [9] **2.1**. *Suppose that* $\sigma : [0,T] \to L_2^0(Y,X)$ *satisfies* $\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathsf{L}_2^0}^2 < \infty$, *and Suppose that* $\{U(t,s), 0 \le s \le t \le T\}$ *is an evolution system of operators satisfying* $\|U(t,s)\| \le Me^{-\beta(t-s)}$, *for some constants* $\beta > 0$ *and* $M \ge 1$ *for all* $t \ge s$. *Then, we have*

$$\mathsf{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 \le CM^2 t^{2H} (\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathsf{L}_2^0})^2.$$

Remark 2.1. By lemma 2.1, the stochastic integral

$$Z(t) = \int_0^t U(t,s)\sigma(s)dB^H(s), \qquad t \in [0,T],$$

is well-defined.

3. Controllability Result

Now we derive the controllability conditions for impulsive neutral evolution stochastic

functional differential equations with variable delays driven by fractional Brownian motion in a real separable Hilbert space. In order to define the concept of mild solution of the problem (1), we consider the following space:
$\mathsf{PC}([-\tau,T],\mathsf{L}^2(\Omega,X)) := \{[-\tau,T] \to \mathsf{L}^2(\Omega,X)$ such that $x(.)$ is continuous except for a finite number of points t_k at which the left and right limits exist and $x(t_k^+) = x(t_k)\}$.
It is easy to verify that PC is a Banach space (see Lemma 2.6 in [31]) with the supremum norm

$$\|\xi\|_{\mathsf{PC}} = \sup_{u\in[-\tau,T]} (\mathsf{E}\|\xi(u)\|^2)^{1/2}.$$

Definition 3.1. An X-valued process $\{x(t),\ t \in [-\tau,T]\}$, is called a mild solution to equation (1) if
i) $x(.) \in \mathsf{PC}([-\tau,T],\mathsf{L}^2(\Omega,X))$,
ii) $x(t) = \varphi(t),\ -\tau \le t \le 0$.
iii) For arbitrary $t \in [0,T]$, we have

$$\begin{aligned} x(t) &= U(t,0)(\varphi(0) + g(0,\varphi(-r(0)))) - g(t,x(t-r(t))) \\ &\quad - \int_0^t U(t,s)A(s)g(s,x(s-r(s)))ds + \int_0^t U(t,s)[f(s,x(s-\rho(s))) + (Bu)(s)]ds \\ &\quad + \int_0^t U(t,s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} U(t,t_k)I_k x(t_k^-), \quad \mathsf{P}-a.s. \end{aligned} \tag{6}$$

Definition 3.2. The impulsive neutral evolution stochastic functional differential equation (1) is said to be controllable on the interval $[-\tau,T]$, if for every initial stochastic process φ defined $[-\tau,0]$ and $x_1 \in X$, there exists a stochastic control $u \in L^2([0,T],U)$ such that the mild solution $x(.)$ of (1) satisfies $x(T) = x_1$.

We will study problem (1) under the following assumptions:
($\aleph$.1) i) The evolution family is exponentially stable, that is, there exist two constants $\beta > 0$ and $M \ge 1$ such that

$$\|U(t,s)\| \le Me^{-\beta(t-s)}, \quad for\,all \quad t \ge s,$$

ii) There exist a constant $M_* > 0$ such that

$$\|A^{-1}(t)\| \le M_* \quad for\ all \quad t \in [0,T].$$

($\aleph$.2) The maps $f,g : [0,T] \times X \to X$ are continuous functions and there exist two positive constants C_1 and C_2, such that for all $t \in [0,T]$ and $x,y \in X$:
i) $\|f(t,x) - f(t,y)\| \vee \|g(t,x) - g(t,y)\| \le C_1\|x-y\|$.
ii) $\|f(t,x)\|^2 \vee \|A^k(t)g(t,x)\|^2 \le C_2(1 + \|x\|^2), \quad k = 0,1$.
($\aleph$.3) i) There exists a positive constant L_* such that

$$\|A(t)g(t,x) - A(t)g(t,y)\| \le L_*\|x-y\|,$$

for all $t \in [0,T]$ and $x,y \in X$.
ii) The function g is continuous in the quadratic mean sense: for all $x(.) \in \mathsf{PC}([-\tau,T],\mathsf{L}^2(\Omega,X))$, we have

$$\lim_{t\to s} \mathsf{E}\|g(t,x(t)) - g(s,x(s))\|^2 = 0.$$

($\aleph$.4) i) The map $\sigma : [0,T] \longrightarrow \mathsf{L}_2^0(Y,X)$ is bounded, that is : there exists a positive constant L such that $\|\sigma(t)\|_{\mathsf{L}_2^0(Y,X)} \leq L$ uniformly in $t \in [0,T]$.
ii) Moreover, we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ satisfies $\varphi \in \mathsf{C}([-\tau,0],\mathsf{L}^2(\Omega,X))$.
($\aleph$.5) $I_k \in C(X,X)$ and there exist a finite positive constants α_k, $\widetilde{\alpha}_k$ such that for all $t \in [0,T]$ and $x,y \in X$, we have:

$$\|I_k(x) - I_k(y)\| \leq \alpha_k \|x-y\| \quad \textit{and} \quad \|I_k(x)\|^2 \leq \widetilde{\alpha}_k(1+\|x\|^2),\ k = 1,2,\dots,m.$$

($\aleph$.6) The linear operator W from U into X defined by

$$Wu = \int_0^T U(T,s)Bu(s)ds$$

has an inverse operator W^{-1} that takes values in $L^2([0,T],U)\setminus kerW$, where $kerW = \{x \in L^2([0,T],U),\ \ Wx = 0\}$ (see [18), and there exists finite positive constants M_b, M_w such that $\|B\| \leq M_b$ and $\|W^{-1}\| \leq M_w$.

The main result of this paper is given in the next theorem.

Theorem 3.1. *Suppose that* $(\aleph.1)-(\aleph.6)$ *hold. Then, the system (1) is controllable on* $[-\tau,T]$ *provided that*

$$L_*^2 M_*^2 + M^2 m \sum_{k=1}^{m} \alpha_k < \frac{1}{8}. \tag{7}$$

Proof. Fix $T > 0$ and let $\mathsf{PC} := \mathsf{PC}([-\tau,T],\mathsf{L}^2(\Omega,X)$ be the Banach space of piecewise continuous maps from $[-\tau,T]$ into $\mathsf{L}^2(\Omega,X)$, equipped with the supremum norm $\|\xi\|_{\mathsf{PC}} = \sup\limits_{u\in[-\tau,T]} (\mathsf{E}\|\xi(u)\|^2)^{1/2}$ and let us consider the set

$$S_T = \{x \in \mathsf{PC} : x(s) = \varphi(s), \quad s \in [-\tau,0]\}.$$

S_T is a closed subset of PC provided with the norm $\|.\|_{\mathsf{PC}}$.

Using the hypothesis $(\aleph.6)$ for an arbitrary function $x(.)$, define the stochastic control

$$\begin{aligned} u(t) = W^{-1}\{& x_1 - U(T,0)(\varphi(0) + g(0,\varphi(-r(0)))) + g(T,x(T-r(T))) \\ &+ \int_0^T AU(T,s)g(s,x(s-r(s)))ds - \int_0^T U(T,s)f(s,x(s-\rho(s))ds \\ &- \int_0^T U(T,s)\sigma(s)dB^H(s) - \sum_{0<t_k<T} U(T,t_k)I_k x(t_k^-)\}(t). \end{aligned}$$

We will now show using this control, that the operator ψ on $S_T(\varphi)$ defined by $\psi(x)(t) = \varphi(t)$ for $t \in [-\tau,0]$ and for $t \in [0,T]$,

$$\psi(x)(t) = U(t,0)(\varphi(0) + g(0,\varphi(-r(0)))) - g(t,x(t-r(t))) - \int_0^t U(t,s)A(s)g(s,x(s-r(s)))ds$$

$$+ \int_0^t U(t,s)f(s,x(s-\rho(s)))ds + \int_0^t U(t,v)\sigma(s)dB^H(s) + \sum_{0<t_k<t} U(t,t_k)I_k x(t_k^-)$$

$$+ \int_0^t U(t,v)BW^{-1}\{x_1 - U(T,0)(\varphi(0) + g(0,\varphi(-r(0)))) + g(T,x(T-r(T)))$$

$$+ \int_0^T U(T,s)A(s)g(s,x(s-r(s)))ds - \int_0^T U(T,s)f(s,x(s-\rho(s))ds$$

$$- \int_0^T U(T,s)\sigma(s)dB^H(s) - \sum_{0<t_k<T} U(T,t_k)I_k x(t_k^-)\}dv,$$

has a fixed point. This fixed point is then a solution of (1). Clearly, $\psi(x)(T) = x_1$, which implies that the system (1) is controllable. For better readability, we break down the proof into sequence of steps.

Step 1: ψ is well defined. Let $x \in S_T(\varphi)$ and $t \in [0,T]$, such that $t \neq t_k$, $k = 1,2,\ldots,m.$, we are going to show that each function $\psi(x)(.)$ is continuous in the $\mathsf{L}^2(\Omega,X)$-sense. Let $|h|$ be sufficiently small. Then for any fixed $x \in S_T$, we have

$$\mathsf{E}\|\psi(x)(t+h) - \psi(x)(t)\|^2 \leq 7\mathsf{E}\|(U(t+h,0) - U(t,0))(\varphi(0) + g(0,\varphi(-r(0))))\|^2$$

$$+7\mathsf{E}\|g(t+h,x(t+h-r(t+h))) - g(t,x(t-r(t)))\|^2$$

$$+7\mathsf{E}\|\int_0^{t+h} U(t+h,s)A(s)g(s,x(s-r(s))ds - \int_0^t U(t,s)A(s)g(s,x(s-r(s))ds\|^2$$

$$+7\mathsf{E}\|\int_0^{t+h} U(t+h,s)f(s,x(s-\rho(s)))ds - \int_0^t U(t,s)f(s,x(s-\rho(s)))ds\|^2$$

$$+7\mathsf{E}\|\int_0^{t+h} U(t+h,s)\sigma(s)dB^H(s) - \int_0^t U(t,s)\sigma(s)dB^H(s)\|^2$$

$$+7\mathsf{E}\|\sum_{0<t_k<t+h} U(t+h,t_k)I_k x(t_k^-) - \sum_{0<t_k<t} U(t,t_k)I_k x(t_k^-)\|^2$$

$$+\ 7\mathsf{E}\|\int_0^{t+h} U(t+h,v)BW^{-1}\{x_1 - U(T,0)(\varphi(0)+g(0,\varphi(-r(0))))$$
$$+\ g(T,x(T-r(T)))$$

$$+\ \int_0^T U(T,s)A(s)g(s,x(s-r(s)))ds - \int_0^T U(T,s)f(s,x(s-\rho(s))ds$$

$$-\ \int_0^T U(T,s)\sigma(s)dB^H(s) - \sum\nolimits_{0<t_k<T} Y(T,t_k)I_k x(t_k^-)\}dv$$

$$-\ \int_0^t U(t,v)BW^{-1}\{x_1 - U(T,0)(\varphi(0)+g(0,\varphi(-r(0))))$$
$$+\ g(T,x(T-r(T)))$$

$$+\ \int_0^T U(T,s)A(s)g(s,x(s-r(s)))ds - \int_0^T U(T,s)f(s,x(s-\rho(s))ds$$

$$-\ \int_0^T U(T,s)\sigma(s)dB^H(s) - \sum\nolimits_{0<t_k<T} U(T,t_k)I_k x(t_k^-)\}dv$$

$$=\ 7\sum_{1\le i\le 7}\mathsf{E}\|I_i(t+h)-I_i(t)\|^2.$$

From definition 3.1, we obtain

$$\lim_{h\to 0}\ (U(t+h,0)-U(t,0))(\varphi(0)+g(0,\varphi(-r(0))))\ =\ 0,$$

and from $(\aleph.1)$, we have

$$\|(U(t+h,0)-U(t,0))(\varphi(0)+g(0,\varphi(-r(0))))\| \le Me^{-\beta t}(e^{-\beta h}+1)\|\varphi(0)+g(0,\varphi(-r(0)))\|$$
$$\in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\to 0}\ \mathsf{E}\|I_1(t+h)-I_1(t)\|^2\ =\ 0.$$

Moreover, assumption $(\aleph.2)$ ensures that

$$\lim_{h\to 0}\ \mathsf{E}\|I_2(t+h)-I_2(t)\|^2\ =\ 0.$$

To show that the third term $I_3(h)$ is continuous, we suppose $h > 0$ (similar calculus for $h < 0$). We have

$$\begin{aligned}\|I_3(t+h)-I_3(t)\| \leq & \left\|\int_0^t (U(t+h,s)-U(t,s))A(s)g(s,x(s-r(s)))ds\right\| \\ & + \left\|\int_t^{t+h}(U(t,s)g(s,x(s-r(s)))ds\right\| \\ \leq & I_{31}(h)+I_{32}(h).\end{aligned}$$

By Hölder's inequality, we have

$$\mathsf{E}\|I_{31}(h)\| \leq t\,\mathsf{E}\int_0^t \|U(t+h,s)-U(t+h,s))A(s)g(s,x(s-r(s))\|^2 ds.$$

By definition 3.1, we obtain

$$\lim_{h\to 0}\ (U(t+h,s)-U(t,s))A(s)g(s,x(s-r(s)))\ = 0.$$

From $(\aleph.1)$ and $(\aleph.2)$, it follows that

$$\begin{aligned}\|U(t+h,s)-U(t,s))A(s)g(s,x(s-r(s))\| \leq C_2 Me^{-\beta(t-s)}(e^{-\beta h}+1)\|A(s)g(s,x(s-r(s))\| \\ \in L^2(\Omega).\end{aligned}$$

Then we conclude, by the Lebesgue dominated theorem, that

$$\lim_{h\to 0}\ \mathsf{E}\|I_{31}(h)\|^2 = 0.$$

Similarly as before, by using $(\aleph.1)$ and $(\aleph.2)$, we get

$$\mathsf{E}\|I_{32}(h)\|^2 \leq \frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathsf{E}\|x(s-r(s))\|^2)ds.$$

Thus,

$$\lim_{h\to 0}\ \mathsf{E}\|I_{32}(h)\|^2 = 0.$$

As for the fourth term $I_4(h)$, we suppose $h > 0$ (similar calculus for $h < 0$), to arrive at

$$\begin{aligned}\|I_4(t+h)-I_4(t)\| \leq & \left\|\int_0^t (U(t+h,s)-U(t,s))f(s,x(s-\rho(s)))ds\right\| \\ & + \left\|\int_t^{t+h}(U(t,s)f(s,x(s-\rho(s)))ds\right\| \\ \leq & I_{41}(h)+I_{42}(h).\end{aligned}$$

By Hölder's inequality, we have

$$\mathsf{E}\|I_{41}(h)\| \leq t\,\mathsf{E}\int_0^t \|U(t+h,s)-U(t,s))f(s,x(s-\rho(s))\|^2 ds.$$

Again by exploiting properties of definition 3.1, we obtain

$$\lim_{h\to 0}\ (U(t+h,s)-U(t,s))f(s,x(s-\rho(s)))\ = 0,$$

and

$$\|U(t+h,s)-U(t,s))f(s,x(s-\rho(s))\| \leq Me^{-\beta(t-s)}(e^{-\beta h}+1)\|f(s,x(s-\rho(s))\| \in L^2(\Omega).$$

Then we conclude, by the Lebesgue dominated theorem, that

$\lim_{h\to 0} \mathsf{E}\|I_{41}(h)\|^2 = 0.$

On the other hand, by $(\aleph.1)$, $(\aleph.2)$, and the Hölder's inequality, we have

$$\mathsf{E}\|I_{42}(h)\|^2 \leq \frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathsf{E}\|x(s-r(s))\|^2)ds.$$

Thus

$$\lim_{h\to 0} I_{42}(h) = 0.$$

Now, for the term $I_5(h)$, we have

$$\begin{aligned}\|I_5(t+h)-I_5(t)\| &\leq \|\int_0^t(U(t+h,s)-U(t,s)\sigma(s)dB^H(s)\| \\ &+ \|\int_t^{t+h}U(t+h,s)\sigma(s)dB^H(s)\| \\ &\leq I_{51}(h)+I_{52}(h).\end{aligned}$$

By lemma 2.1, we may write

$$E\|I_{51}(h)\|^2 \leq 2Ht^{2H-1}\int_0^t\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathsf{L}_2^0}ds.$$

Since

$$\lim_{h\to 0}\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathsf{L}_2^0} = 0,$$

and

$$\|(U(t+h,s)-U(t,s)\sigma(s)\|_{\mathsf{L}_2^0} \leq MLe^{-\beta(t-s)}e^{-\beta h+1} \in \mathsf{L}^1([0,T],ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathsf{E}\|I_{51}(h)\|^2 = 0.$$

Again by lemma 2.1, we may write

$$\mathsf{E}\|I_{52}(h)\|^2 \leq \frac{2Ht^{2H-1}LM^2(1-e^{-2\beta h})}{2\beta}.$$

Thus,

$$\lim_{h\to 0}\mathsf{E}\|I_{52}(h)\|^2 = 0.$$

As for the estimation of term I_6, we make use of condition $(\mathsf{H}.5)$, to write

$$\begin{aligned}\mathsf{E}\|I_6(t+h)-I(t)\|^2 &\leq \sum_{0<t_k<T}\mathsf{E}\|I_kx(t_k^-)\|^2\|U(t+h,t_k)-U(t,t_k)\|^2 \\ &\leq m\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\|U(t+h,t_k)-U(t,t_k)\|^2,\end{aligned}$$

then

$$\lim_{h\to 0}\mathsf{E}\|I_6(h)\|^2 = 0.$$

We now use conditions $(\aleph.1)-(\aleph.6)$, to get

$$\begin{aligned}
\mathsf{E}\|I_7(h)\|^2 \leq\ & 2\mathsf{E}\|\int_t^{t+h} U(t+h,v)BW^{-1}\{x_1 - U(T,0)(\varphi(0)+g(0,\varphi(-r(0))))\\
&+ g(T,x(T-r(T))) + \int_0^T U(T,s)A(s)g(s,x(s-r(s)))ds\\
&- \int_0^T U(T,s)f(s,x(s-\rho(s)))ds - \int_0^T U(T,s)\sigma(s)dB^H(s)\\
&- \sum_{0<t_k<T} U(T,t_k)I_k x(t_k^-)\}dv\|^2\\
&+ 2\mathsf{E}\|\int_0^t (U(t+h,v)-U(t,v))BW^{-1}\{x_1 - U(T,0)(\varphi(0)+g(0,\varphi(-r(0))))\\
&+ g(T,x(T-r(T))) + \int_0^T U(T,s)A(s)g(s,x(s-r(s)))ds\\
&- \int_0^T U(T,s)f(s,x(s-\rho(s)))ds - \int_0^T U(T,s)\sigma(s)dB^H(s)\\
&- \sum_{0<t_k<T} U(T,t_k)I_k x(t_k^-)\}dv\|^2\\
\leq\ & 2[\mathsf{E}\|I_{7,1}(h)\|^2 + \mathsf{E}\|I_{7,2}(h)\|^2].
\end{aligned}$$

Let's first deal with $I_{7,1}(h)$; it follows from the conditions $(\aleph.1)-(\aleph.6)$ that

$$\begin{aligned}
\mathsf{E}\|I_{7,1}(h)\|^2 \leq\ & 7M^2M_b^2M_w^2\int_t^{t+h}\{\mathsf{E}\|x_1\|^2 + M^2\mathsf{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2\\
&+ M_*^2C_2T(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2) + M^2TC_2(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\\
&+ M^2TC_2(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2) + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathsf{L}_2^0}ds\\
&+ M^2m\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\}dv.
\end{aligned}$$

This results with

$\lim\limits_{h\to 0} \mathsf{E}\|I_{7,1}(h)\|^2 = 0.$

In a similar fashion, we have

$$\begin{aligned}\mathsf{E}\|I_{7,2}(h)\|^2 \quad &\leq 7M_b^2M_w^2\int_0^t\|U(t+h,v)-U(t,v)\|^2\{\mathsf{E}\|x_1\|^2\\ &+M^2\mathsf{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2+M_*^2C_2(1+\mathsf{E}\|x\|^2)\\ &+M^2T^2C_2(1+\mathsf{E}\|x\|^2)+M^2T^2C_2(1+\mathsf{E}\|x\|^2)\\ &+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathsf{L}_2^0}ds+M^2m\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup\nolimits_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\}dv.\end{aligned}$$

Since

$$\begin{aligned}&\|U(t+h,v)-U(t,v)\|^2\{\mathsf{E}\|x_1\|^2+M^2\mathsf{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2\\ &+M_*^2C_4(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\\ &+M^2T^2C_2(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)+M^2T^2C_2(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\\ &+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathsf{L}_2^0}ds+M^2m\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\}\\ &\leq 4M^2\{\mathsf{E}\|x_1\|^2+M^2\mathsf{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2\\ &+M_*^2C_2(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)+2M^2T^2C_2(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\\ &+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathsf{L}_2^0}ds+M^2m\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2)\}\quad\in L^1([0,T],ds]),\end{aligned}$$

we conclude, by the dominated convergence theorem, that,

$\lim\limits_{h\to 0} \mathsf{E}\|I_{7,2}(h)\|^2 = 0.$

The above arguments show that $\lim\limits_{h\to 0}\mathsf{E}\|\psi(x)(t+h)-\psi(x)(t)\|^2 = 0$. Hence, we conclude that the function $t \to \psi(x)(t)$ is continuous in the L^2-sense.

Step 2: Now, we are going to show that ψ is a contraction mapping in $S_{T_1}(\varphi)$ with some $T_1 \leq T$ to be specified later. Let $x, y \in S_T(\varphi)$, then for any fixed $t \in [0, T]$, we have

$$\begin{aligned}
&\mathsf{E}\ \| \ \psi(x)(t) - \psi(y)(t)\|^2 \\
&\leq 8\|A(t)^{-1}\|^2 \mathsf{E}\|A(t)g(t,x(t-r(t)))\ - A(t)g(t,y(t-r(t)))\|^2 \\
&+ 8\mathsf{E}\|\int_0^t U(t,s)A(s)(g(s,x(s-r(s)))\ - g(s,y(s-r(s))))ds\|^2 \\
&+ 8\mathsf{E}\|\int_0^t U(t,s)(f(s,x(s-\rho(s)))\ - f(s,y(s-\rho(s))))ds\|^2 \\
&+ 8\mathsf{E}\|\sum_{0<t_k<T} U(T,t_k)(I_k x(t_k^-) - I_k y(t_k^-))\|^2 \\
&+ 8\mathsf{E}\|\int_0^t U(t,v)BW^{-1}[g(T,x(T-r(T)))\ - g(T,y(T-r(T)))]dv\|^2 \\
&+ 8\mathsf{E}\|\int_0^t U(t,v)BW^{-1}\int_0^T U(T,s)A(s)[g(s,x(s-r(s)))\ - g(s,y(s-r(s)))]ds]dv\|^2 \\
&+ 8\mathsf{E}\|\int_0^t U(t,v)BW^{-1}\int_0^T U(T,s)[f(s,x(s-\rho(s)))\ - f(s,y(s-\rho(s)))]dsdv\|^2 \\
&+ 8\mathsf{E}\|\int_0^t U(t,v)BW^{-1}\sum_{0<t_k<T} U(T,t_k)[I_k x(t_k^-) - I_k y(t_k^-)]dv\|^2.
\end{aligned}$$

By assumptions $(\aleph.1) - (\aleph.6)$ combined with Hölder's inequality, we get that

$$\begin{aligned}
\mathsf{E}\|\psi(x)(t) - \psi(y)(t)\|^2 \ &\leq 8L_*^2 M_*^2 \sup_{s\in[-\tau,\, t]} \mathsf{E}\|x(t-r) - y(t-r)\|^2 \\
&+ 8M^2 L_*^2 \frac{1-e^{-2\beta t}}{2\beta} t \sup_{s\in[-\tau,\, t]} \mathsf{E}\|x(s) - y(s)\|^2
\end{aligned}$$

$$+8M^2C_1^2\frac{1-e^{-2\beta t}}{2\beta}t\sup_{s\in[-\tau,t]}\sup_{s\in[-\tau,t]}\mathsf{E}\|x(s)-y(s)\|^2$$

$$+8M^2m\sum_{k=1}^{m}\alpha_k\sup_{s\in[-\tau,t]}\mathsf{E}\|x(s)-y(s)\|^2$$

$$+8t\,M^2M_b^2M_w^2[C_1^2\mathsf{E}\|x(T-r(T))\ -y(T-r(T))\|^2$$

$$+L_*^2M^2T^2\sup_{s\in[-\tau,t]}\mathsf{E}\|x(s-r(s))\ -y(s-r(s))\|^2$$

$$+T^2M^2C_1^2\sup_{s\in[-\tau,t]}\mathsf{E}\|x(s)-y(s)\|^2$$

$$+M^2m\sum_{k=1}^{m}\alpha_k\sup_{s\in[-\tau,t]}\mathsf{E}\|x(s)-y(s)\|^2].$$

Hence

$$\sup_{s\in[-\tau,t]}\mathsf{E}\|\psi(x)(s)-\psi(y)(s)\|^2\leq\gamma(t)\sup_{s\in[-\tau,t]}\mathsf{E}\|x(s)-y(s)\|^2,$$

where

$$\gamma(t)\ =\ 8[\|L_*^2M_*^2+M^2L_*^2\frac{1-e^{-2\beta t}}{2\beta}t+M^2C_1^2\frac{1-e^{-2\beta t}}{2\beta}t+M^2m\sum_{k=1}^{m}\alpha_k$$

$$t\,M^2M_b^2M_w^2(C_1^2+L_*^2M^2T^2+T^2M^2C_1^2+M^2m\sum_{k=1}^{m}\alpha_k)].$$

By condition (7), we have $\gamma(0)=8(L_*^2M_*^2+M^2m\sum_{k=1}^{m}\alpha_k)<1$. Then there exists $0<T_1\leq T$ such that $0<\gamma(T_1)<1$ and ψ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (1) on $[-\tau,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. Clearly, $(\psi x)(T)=x_1$ which implies that the system (1) is controllable on $[-\tau,T]$. This completes the proof. ■

3. An Example

In recent years, the interest in impulsive neutral systems has been growing rapidly due to their successful applications in practical fields such as physics, chemical technology, bioengineering, and electrical networks. Here consider the following stochastic impulsive neutral functional differential equation with finite delays τ_1 and τ_2 $(0 \le \tau_i \le \tau < \infty,\ i = 1,2)$:

$$\begin{cases} d[u(t,\zeta) + g_1(t,u(t-\tau_1,\zeta))] = [\frac{\partial^2}{\partial^2\zeta}u(t,\zeta) + b(t,\zeta)u(t,\zeta) + f_1(t,u(t-\tau_2,\zeta)) \\ \qquad + \ v(t,\xi)]dt + \sigma(t)dB^H(t),\ 0 \le t \le T,\ 0 \le \zeta \le \pi, \\ \Delta x(t_k,\xi) = x(t_k^+,\xi) - x(t_k^-,\xi) = \frac{C}{k^2}x(t_k,\xi), \quad t = t_k,\ c \ge 0, k = 1,2,\ldots,m, \\ u(t,0) = u(t,\pi) = 0, \qquad 0 \le t \le T, \\ u(t,\zeta) = \varphi(t,\zeta), \qquad t \in [-\tau,0],\ 0 \le \zeta \le \pi, \end{cases} \tag{8}$$

where B^H is a fractional Brownian motion, $b(t,\zeta)$ is a continuous function and is uniformly Hölder continuous in t, f_1, $g_1 : \mathrm{R}^+ \times \mathrm{R} \to \mathrm{R}$ are continuous functions.

To study this system, we consider the space $X = L^2([0,\pi])$ and the operator $A : D(A) \subset X \to X$, given by $Ay = y''$ with,

$$D(A) = \{y \in X : y'' \in X, \quad y(0)) = y(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathrm{N}$ and the corresponding normalized eigenfunctions given by

$$e_n := \sqrt{\frac{2}{\pi}}\sin nx,\ n = 1,2,\ldots.$$

In addition $(e_n)_{n\in\mathrm{N}}$ is a complete orthonormal basis in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2t} < x, e_n > e_n,$$

for $x \in X$ and $t \ge 0$.

Now, we define an operator $A(t) : D(A) \subset X \to X$ by

$$A(t)x(\zeta) = Ax(\zeta) + b(t,\zeta)x(\zeta).$$

By assuming that $b(.,.)$ is continuous and that $b(t,\zeta) \le -\gamma$ $(\gamma > 0)$ for every $t \in \mathrm{R}$, $\zeta \in [0,\pi]$, it follows that the system

$$\begin{cases} u'(t) = A(t)u(t), \quad t \ge s, \\ u(s) = x \in X, \end{cases}$$

has an associated evolution family given by

$$U(t,s)x(\zeta) = \left[T(t-s)\exp(\int_s^t b(\tau,\zeta))d\tau)x \right](\zeta).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that, for every

$s,t \in [0,T]$ with $t > s$,

$\|U(t,s)\| \le e^{-(\gamma+1)(t-s)}$.

In addition, $A(t)$ satisfies the assumption $(\aleph.1)$ (see [3,25]).

To rewrite the initial-boundary value problem (8) in abstract form we assume the following:

i) $B : U \longrightarrow X$ is a bounded linear operator defined by

$Bu(t)(\xi) = v(t,\xi),\ 0 \le \xi \le \pi, u \in L^2([0,T],U)$.

ii) The operator $W : L^2([0,T],U) \to X$, defined by

$$Wu = \int_0^T S(T-s)v(t,\xi)ds,$$

has an inverse W^{-1} and satisfies condition $(\aleph.6)$. For the construction of the operator W and its inverse, see [30].

iii) The substitution operator $f : [0,T] \times X \longrightarrow X$, defined by $f(t,u)(.) = f_1(t,u(.))$, is continuous and we impose suitable conditions on f_1 to verify assumption $(\aleph.2)$.

iv) The substitution operator $g : [0,T] \times X \longrightarrow X$, defined by $g(t,u)(.) = g_1(t,u(.))$, is continuous and we impose suitable conditions on g_1 to verify assumptions $(\aleph.2)$ and $(\aleph.3)$.

If we put

$$\begin{cases} x(t)(\zeta) = x(t,\zeta),\ t \in [0,T], \zeta \in [0,\pi] \\ x(t,\zeta) = \varphi(t,\zeta),\ t \in [-\tau,0],\ \zeta \in [0,\pi], \end{cases} \tag{9}$$

then, the problem (9) can be written in the abstract form

$$\begin{cases} d[x(t) + g(t,x(t-r(t)))] = [A(t)x(t) + f(t,x(t-\rho(t))) + Bu(t)]dt + \sigma(t)dB^H(t), \\ t \in [0,T], t \neq t_k \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k x(t_k^-), \quad t = t_k,\ k = 1,2,\ldots,m, \\ x(t) = \varphi(t), \ -\tau \le t \le 0 \quad a.s. \quad \tau > 0, \end{cases}$$

Furthermore, if we assume that the initial data $\varphi = \{\varphi(t) : -\tau \le t \le 0\}$ satisfies $\varphi \in \mathsf{C}([-\tau,0],\mathsf{L}^2(\Omega,X))$, thus all the assumptions of theorem 3.1 are fulfilled. Therefore, we may conclude that the system (8) is controllable on $[-\tau,T]$.

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