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Solution of Singular Linear Vibrational BVPs by the Homotopy Analysis Method

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Abstract. *In this paper, the homotopy analysis method (HAM) which is a powerful technique of nonlinear analysis, is examined in the solution of the linear vibration equation for very a large circular membrane. The method is shown to be able to handle a variety of pertaining singular boundary value problems (BVPs), to allow for graphical presentation of the results, and appears to be a viable alternative to applicable integral transform methods.*

Key words : Homotopy analysis method ; Linear vibration equation ; Singular boundary value problems; Series solution.

AMS Subject Classifications : 65H20, 65N99, 65Z05

1. Introduction

Nonlinear differential equations are knwn to arise from mathematical modeling of many physics and engineering problems. In most cases, analytical solutions to these equations are either quite difficult or impossible to achieve. Perturbation methods represent the most versatile tools available in nonlinear analysis of engineering problems and provide for analytical approximate solutions that can be superior to numerical solutions. Almost all perturbation methods are, however, based on an assumption that a small parameter exists in the differential equation. Additionally, even when a suitable small parameter exists, the approximate solutions obtained by these methods are valid only for small values of this parameter [4, 13]. Moreover, many nonlinear problems do not contain such perturbable quantities and hence call for the use of non-perturbation methods, such as the artificial small parameter method [12], the δ-expansion method [6] and the Adomian's decomposition method [2]. However, both the perturbation and non-perturbation methods cannot yield a simple way to adjust and control the convergence rate and/or domain of a pertaining approximate series. The homotopy analysis method [7, 8] has been developed to yield accurate asymptotic solutions to nonlinear problems and has successfully been applied to many nonlinear problems such as nonlinear vibrations [11], nonlinear water waves [9], viscous flows of non-Newtonian fluids [10, 15, 16], nonlinear boundary flow and heat transfer [17], [19], von Kármán viscous flow [18], nonlinear fractal Riccati differential equations [3], Black-Scholes equations [20, 21] and others. Liao has proved , in particular, that the homotopy analysis method embraces logically some other non-perturbation techniques, such as Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method. Hayat et al. [5], Sajid et al. [14] and Abbasbandy [1] have illustrated that the homotopy perturbation method is only a special case of the homotopy analysis method.

The study of the wave equation for very large circular membrane, of radius R, $(R \to \infty)$ is fundamental in the design of drums, receivers and loudspeakers [22], and is widely known to be feasible by an asymptotic version of the method of separation of variables or by means of Hankel transforms. Our aim in this paper is, however, to examine the homotopy analysis method as an alternative means for solving this singular BVP :

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \; r \geq 0, \; t \geq 0, \tag{1}$$

subject to the initial conditions

$$u(r,0) = f(r), \;\; \frac{\partial}{\partial t}u(r,0) = cg(r), \tag{2}$$

with $u(r,t)$ representing the space-time displacement, and c is the wave speed of free vibration. Here $f(r)$ and $g(r)$ can obviously represent a variety of forms for the previous BVP.

2. The Method

To summarize the basic concepts of the homotopy analysis method (HAM), we consider a general nonlinear equation,

$$N[u(r,t)] = 0,$$

with N as a nonlinear operator, $u(r,t)$ is the unknown function, and r and t denote the same spatial and temporal variables of (1)-(2). The initial conditions (2) will enter into the picture at a later stage. By generalizing the traditional homotopy method, Liao in [8] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(r,t;p) - u_0(r,t)] = p\hbar\mathcal{H}(r,t)\mathcal{N}[\phi(r,t;p)], \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(r,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, and $\phi(r,t;p)$ is an unknown function. Interestingly, there is a marked great freedom in the choice of auxiliary parameters in the HAM. Obviously, when $p = 0$ and $p = 1$,

$$\phi(r,t;0) = u_0(r,t), \;\; \phi(r,t;1) = u(r,t),$$

respectively hold. Thus as p increases from 0 to 1, the solution $\phi(r,t;p)$ varies from the initial guess $u_0(r,t)$ to the solution $u(r,t)$. Furthermore, by expanding $\phi(r,t;p)$ in Taylor series with respect to p, one gets

$$\phi(r,t;p) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t)p^m, \tag{4}$$

where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \phi(r,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{5}$$

If the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter $\hbar$ and the auxiliary function ϕ are all properly chosen, then the series (4) converges, when $p = 1$, to end up with

$$u(r,t) = \phi(r,t;1) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \tag{6}$$

Hence, according to the definition (5), the governing equation for $u_m(r,t)$ can be deduced from the zero-order deformation equation (3) as follow. Define the vector

$$\vec{u}_n = \{u_0(r,t), u_1(r,t), \ldots, u_n(r,t)\},$$

then differentiate Eq. (3) m times with respect to the embedding parameter p. Finally, by setting $p = 0$ and dividing them by $m!$ we arrive at the so-called mth-order deformation equation

$$\mathcal{L}[u_m(r,t) - \chi_m\, u_{m-1}(r,t)] = \hbar\, \mathcal{H}(r,t)\, R_m(\vec{u}_{m-1}), \tag{7}$$

in which

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(r,t;p)]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{8}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $u_m(r,t)$ for $m \ge 1$ is governed by the linear equation (7) subject to the linear initial conditions that come from original problem (1)-(2). This can easily be solved by symbolic computation program softwares such as Maple or Mathematica.

Theorem (Convergence) [8]. Let $u_m(r,t)$ be a solution to the high-order deformation equation (7) in the context of (5)-(6). As long as the series (6),

$$u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t)$$

is convergent, it must be a solution to the original nonlinear equation $N[u(r,t)] = 0$.
Proof. Is reported in [8], page 57.

3. Application

Rewrite the previous linear variable coefficient partial differential singular BVP as

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = 0,\ r \geq 0,\ t \geq 0,\\
&\qquad u(r,0) = f(r),\\
&\qquad \frac{\partial}{\partial t}u(r,0) = cg(r).
\end{aligned} \tag{9}$$

According to (9), we choose the linear operator

$$\mathcal{L}[\phi(r,t;p)] = \frac{\partial^2 \phi(r,t;p)}{\partial t^2},$$

with the property

$$\mathcal{L}[c_1(r) + c_2(r)t] = 0,$$

where $c_1(r)$ and $c_2(r)$ are functions with respect to r. We shall then conceive the linear variable coefficient operator

$$\frac{\partial^2 \phi(r,t;p)}{\partial t^2} - c^2\left(\frac{\partial^2 \phi(r,t;p)}{\partial r^2} + \frac{1}{r}\frac{\partial \phi(r,t;p)}{\partial r}\right), \tag{10}$$

metaphorically as the nonlinear operator $\mathcal{N}[\phi(r,t;p)]$. It is obvious then that the initial approximation should be in the form $u_0(r,t) = f(r) + cg(r)t$, which satisfies the initial conditions. We also choose $H(r,t) = 1$ for simplicity. Hence from (3), the general zero-order deformation equation is

$$(1-p)\mathcal{L}[\phi(r,t;p) - u_0(r,t)] = p\hbar\mathcal{N}[\phi(r,t;p)], \tag{11}$$

with the initial conditions

$$\begin{aligned}
&\phi(r,0;p) = f(r),\\
&\frac{\partial}{\partial t}\phi(r,0,p) = cg(r).
\end{aligned} \tag{12}$$

The associated high-order deformation equation is

$$\mathcal{L}[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar R_m(\vec{u}_{m-1}), \tag{13}$$

with the initial conditions

$$\begin{aligned}
&u_m(r,0) = 0,\\
&u'_{t,m}(r,0) = 0.
\end{aligned} \tag{14}$$

From (8) and (10), we have:

$$R_m(\vec{u}_{m-1}) = u''_{t,m-1}(r,t) - c^2[u''_{r,m-1}(r,t) + \frac{1}{r}u'_{r,m-1}(r,t)], \tag{15}$$

where the prime with indexes r and t denote differentiation with respect to r and t, respectively. Now, the solution of the mth-order deformation Eq. (13) for $m \geq 1$, when $R_m(\vec{u}_{m-1})$ is defined by (15), becomes

$$u_m(r,t) = \chi_m u_{m-1}(r,t) + \hbar\int_0^t\int_0^s R_m(\vec{u}_{m-1})(r,\tau)d\tau ds + c_1(r) + c_2(r)t, \tag{16}$$

where $c_1(r)$ and $c_2(r)$ are determined by the initial conditions (14). Now, we can obtain Mth-order approximate solution $U_M(r,t) = \sum_{m=0}^{M} u_m(r,t)$, by computing the u_i's from (16) successively. Finally by choosing proper value for $\hbar$ from valid region, which is obtained from the so-called $\hbar$-curve , see Liao in [8], the construction of the approximate solution completes.

4. Results

In this section we analyze the singular BVP (9) for the illustrative example of particular initial conditions and choosing proper $\hbar$ with the help of $\hbar$-curves. This allows us finally to present the results in a graphical fashion.
Let us consider then

$$f(r) = r^2, \ \ g(r) = \frac{1}{r},$$

together with the linear partial differential equation of (9). Following the steps described by (10)-(16) , we obtain the first few approximations of the HAM as follows

$$\begin{aligned} U_0(r,t) &= r^2 + c\frac{1}{r}t, \\ U_1(r,t) &= r^2 + c\frac{1}{r}t + h\left(-\frac{1}{6}\,\frac{c^3t^3}{r^3} - 2c^2t^2\right), \\ U_2(r,t) &= r^2 + \frac{ct}{r} - 4c^2ht^2 - 2c^2h^2t^2 - \frac{c^3ht^3}{3\,r^3} - \frac{c^3h^2t^3}{6r^3} + \frac{3c^5h^2t^5}{40r^5}. \end{aligned} \tag{17}$$

To assess the impact of the value of $\hbar$ on the convergence of (17), we first plot the so-called $\hbar$-curves of $U''_{30,t}(1,0)$ and $U'''_{30,t}(1,0)$ in Fig. 1 (notice that $U'_{30,t}(r,0)$ is not very useful because it is only a constant). From this figure, it is easy to discover the valid domain of $\hbar$ ($\hbar \in [-1.8, -0.2]$) for the present BVP. Clearly HAM can provide us with a convenient way to adjust and control the convergence rate and domain for the pertaining approximation series to the solution of the posing singular BVP.

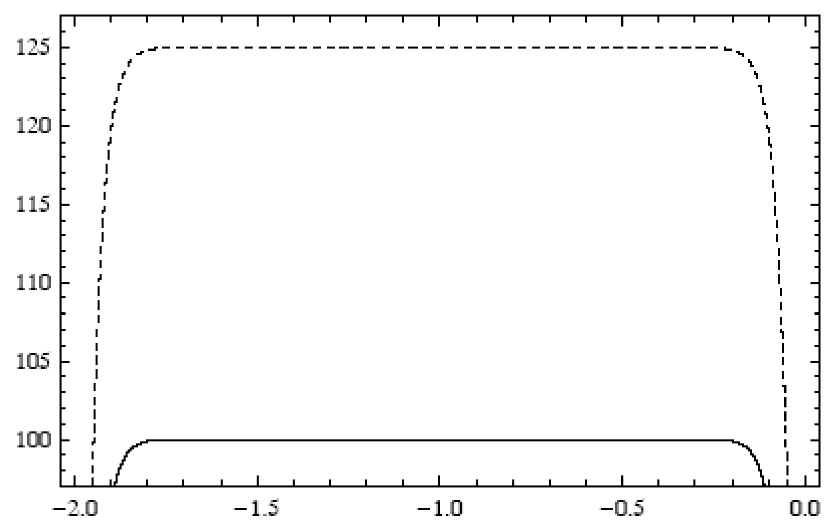

Fig. 1: The $\hbar$-curves with $c = 5$ for 30th-order approximation; Bold line: $U''_{30,t}(1,0)$; Dotted line: $U'''_{30,t}(1,0)$.

Let us choose $\hbar = -1$ for simplicity, for this value of $\hbar$, the 30th-order approximate solution has been plotted in Fig. 2. As we see, the displacement increases with the increase of r and t. It is also seen from Fig. 3 that the displacement increases fast with the increase both of t and c at a fixed value of the radius of the membrane.

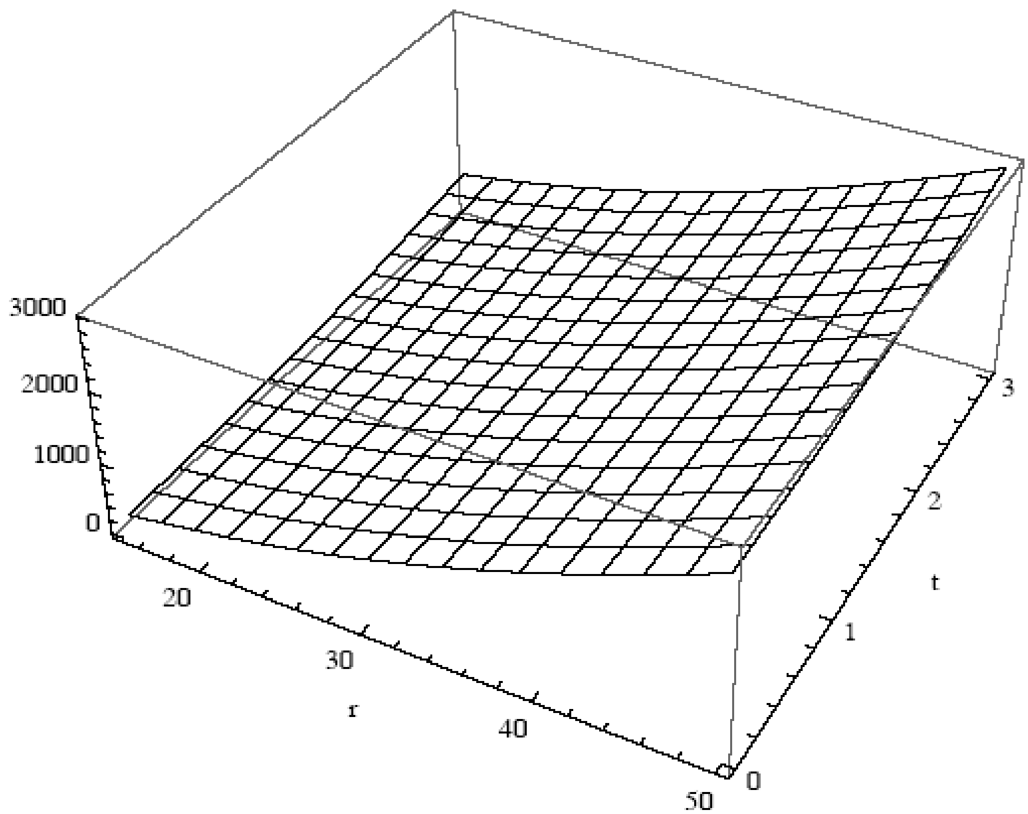

Fig. 2: $U_{30}(r,t)$ with respect to r and t for $c = 5$ and $\hbar = -1$.

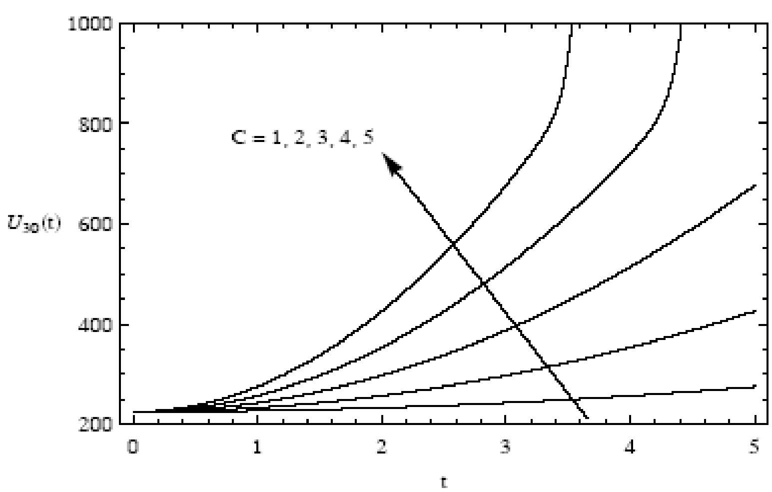

Fig. 3: $U_{30}(r,t)$ with respect to t for $r = 15$ and different c's.

5. Conclusions

In this paper, the homotopy analysis method (HAM) has been demonstrated to be applicable in the solution of the linear variable coefficient vibrational singular BVP. Equipped with a flexibility in choosing $\hbar$, the HAM exhibits a unique feature for controlling the convergence of the approximation series to the solution of this problem. This is a further illustration of the universal applicability of the homotopy method to the analysis of both linear and nonlinear BVPs and its viable role for linear BVPs as an alternative to separation of variables and/or integral transform methods.

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