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On General Inexact Proximal Point Algorithms and Their Contributions to Linear Convergence Analysis

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Abstract. *Based on the notion of relative maximal monotonicity, a hybrid proximal point algorithm is introduced and then it is applied to the approximation solvability of a general class of variational inclusion problems, while achieving a linear convergence. The obtained results generalize the celebrated work of Rockafellar (1976) where the Lipschitz continuity at 0 of the inverse of the set-valued mapping is considered.*

Key words: Inclusion problems, Maximal monotone mapping, Relative maximal monotone mapping, Generalized resolvent operator.

AMS Subject Classifications : 49J40, 47H10, 65B05

1. Introduction

Let X be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and with the norm $\| \cdot \|$ on X. We consider the inclusion problem: Determine a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.
In ([5, Theorem 2), Rockafellar investigated the general convergence of the proximal point algorithm in the context of solving (1), by showing for M maximal monotone, that the sequence $\{x^k\}$ generated for an initial point x^0 by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{2}$$

converges strongly to a solution of (1), provided the approximation is made sufficiently accurate as the iteration proceeds, where $P_k = (I + c_k M)^{-1}$ is the resolvent operator for a sequence $\{c_k\}$ of positive real numbers, that is bounded away from zero. We observe from (2) that x^{k+1} is an approximate solution to inclusion problem

$$0 \in M(x) + c_k^{-1}(x - x^k). \tag{3}$$

We state the theorem of Rockafellar ([5], Theorem 2) for the sake of the completeness. This work became very significant because the approach of the Lipschitz continuity of M^{-1} instead of applying the nonexpansiveness of the resolvent turned out to be quite remarkable.

Theorem 1.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal monotone. For an arbitrarily chosen initial point $x^0,$ let the sequence $\{x^k\}$ be generated by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{4}$$

such that

$$\|x^{k+1} - P_k(x^k)\| \leq \epsilon_k,$$

where $P_k = (I + c_k M)^{-1}$, and the scalar sequences $\{\epsilon_k\}$ and $\{c_k\},$ respectively, satisfy $\Sigma_{k=0}^{\infty}\epsilon_k < \infty$ and $\{c_k\}$ is bounded away from zero.
We further suppose that sequence $\{x^k\}$ is generated by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{5}$$

such that

$$\|x^{k+1} - P_k(x^k)\| \leq \delta_k \|x^{k+1} - x^k\|,$$

where scalar sequences $\{\delta_k\}$ and $\{c_k\},$ respectively, satisfy $\Sigma_{k=0}^{\infty}\delta_k < \infty$ and $c_k \uparrow c \leq \infty.$
Also, assume that $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1), and that M^{-1} is (a) –Lipschitz continuous at 0 for a>0. Let

$$\mu_k = \frac{a}{\sqrt{a^2 + c_k^2}} < 1.$$

Then the sequence $\{x^k\}$ converges strongly to $x^*,$ a unique solution to (1) with

$$\|x^{k+1} - x^*\| \leq \alpha_k \|x^k - x^*\| \ \forall\, k \geq k', \tag{6}$$

where

$$0 \leq \alpha_k = \frac{\mu_k + \delta_k}{1 - \delta_k} < 1 \ \forall\, k \geq k', \tag{7}$$

and

$$\alpha_k \to 0 \ as\ c_k \to \infty. \tag{8}$$

We observe that most the of variational problems, including minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, minimax problems, and decision and management sciences can be unified into form (1), and approximation solvability is achievable using some sort of resolvent techniques based on suitable algorithmic procedures. However, general maximal monotonicity has played a crucial role by providing a powerful framework to develop and use suitable proximal point algorithms investigating convex programming and variational inequalities. This algorithm turned out to be of more interest because of its role in certain computational methods based on duality, such as the method of multipliers in nonlinear programming. For more details, we refer the reader [1-15].

In this communication, we first derive some auxiliary results on relatively maximal monotone and cocoercive mappings, and then examine the approximation solvability of variational inclusion problem (1). Our approach for the solvability of (1) differs than that of [5] in the sense that M is without monotonicity assumption, there is no assumption of the Lipschitz continuity on M^{-1}, and the proof turns out to be simple and compact. Note that this new model collapses when M is not relatively maximal monotone.

2. Relative Maximal Monotonicity

In this section, first we introduce the notion of the *relative maximal monotonicity*, and then we discuss some basic properties along with some auxiliary results for the problem on hand. It just happened that this notion sounds more application-oriented in the sense of applications to variational inclusion problems.
Let X be a real Hilbert space with the norm $\|\cdot\|$ for X, and with the inner product $\langle\cdot,\cdot\rangle$.

Definition 2.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be a multivalued mapping and $A : X \to X$ be a single-valued mapping on X. The map M is said to be:
(i) Monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0 \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

(ii) Strictly monotone if M is monotone and equality holds only if $u = v$.
(iii) (r) – strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*,\ u - v\rangle \geq r\|u - v\|^2 \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

(iv) (r) – expanding if there exists a positive constant r such that

$$\|u^* - v^*\| \geq r\|u - v\| \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

(v) (m) –cocoercive if there exists a positive constant m such that

$$\langle u^* - v^*,\ u - v\rangle \geq m\|u^* - v^*\|^2 \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

(vi) Monotone with respect to A if

$$\langle u^* - v^*, A(u) - A(v)\rangle \ \geq 0 \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

(vii) Strictly monotone with respect to A if M is monotone with respect to A and equality holds

only if $u = v$.
(viii) (r) – strongly monotone with respect to A if there exists a positive constant r such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \ \geq r\|u - v\|^2 \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

(ix) (m) –cocoercive with respect to A if there exists a positive constant m such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \ \geq m\|u^* - v^*\|^2 \,\forall\, (u,u^*),(v,v^*) \in graph(M).$$

Definition 2.2. Let X be a real Hilbert space, and let $M : X \to 2^X$ be a mapping on X. Furthermore, let $A : X \to X$ be a single-valued mapping on X. The map M is said to be:
(i) Nonexpansive if

$$\|u^* - v^*\| \leq \|u - v\| \; \forall (u, u^*), (v, v^*) \in graph(M).$$

(ii) Cocoercive if

$$\langle u^* - v^*, u - v\rangle \geq \|u^* - v^*\|^2 \leq\rangle \; \forall (u, u^*), (v, v^*) \in graph(M).$$

(iii) Cocoercive with respect to A if

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq \|u^* - v^*\|^2 \; \forall (u, u^*), (v, v^*) \in graph(M).$$

Definition 2.3. Let X be a real Hilbert space. Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be relative maximal monotone if
(i) M is monotone with respect to A, that is,

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \; \forall (u, u^*), (v, v^*) \in graph(M),$$

(ii) $R(I + \rho M) = X$ for $\rho > 0$.

Definition 2.4. Let X be a real Hilbert space. Let $A : X \to X$ be an (r) −strongly monotone mapping, and let $M : X \to 2^X$ be a relative maximal monotone mapping. Then the generalized resolvent operator $R^M_{\rho,A} : X \to X$ is defined by

$$R^M_{\rho,A}(u) = (I + \rho M)^{-1}(u).$$

Proposition 2.1. Let X be a real Hilbert space. Let $A : X \to X$ be an (r) −strongly monotone mapping, and let $M : X \to 2^X$ be a relative maximal monotone mapping. Then the operator $R^M_{\rho,A} = (I + \rho M)^{-1}$ is single-valued.

Proof. The proof follows from the definition of the resolvent operator. ■

Definition 2.5. Let X be a real Hilbert space. A map $M : X \to 2^X$ is said to be maximal monotone if
(i) M is monotone, that is,

$$\langle u^* - v^*, u - v\rangle \geq 0 \; \forall (u, u^*), (v, v^*) \in graph(M),$$

(ii) $R(I + \rho M) = X$ for $\rho > 0$.

Definition 2.6. Let X be a real Hilbert space. Let $M : X \to 2^X$ be a maximal monotone mapping. Then the resolvent operator $J^M_\rho : X \to X$ is defined by

$$J^M_\rho(u) = (I + \rho M)^{-1}(u).$$

Next, we include some examples on the relative monotonicity, especially to the context of Definition 2.3.

Example 2.1. Let $X = (-\infty, +\infty)$, $A(x) = -\frac{1}{2}x$ and $M(x) = -x$ for all $x \in X$. Then M is relative monotone but not monotone.

Example 2.2. Let X be a real Hilbert space, and let map $M : X \to 2^X$ be maximal monotone. Suppose that $M_\rho = \rho^{-1}(I - J_\rho^M)$ is the Yosida approximation corresponding to Definition 2.5. Since $M_\rho(u) \in M(J_\rho^M)(u),$ it follows that M_ρ is monotone with respect to $J_\rho^M,$ that is,

$$\langle M_\rho(u) - M_\rho(v), J_\rho^M(u) - J_\rho^M(v)\rangle \ \geq 0 \, \forall \, u, v \in X.$$

3. Generalizations

This section deals with a generalization to Rockafellar's theorem ([5], Theorem 2) under the framework of relative maximal monotonicity, while solving (1).

Theorem 3.1. *Let X be a real Hilbert space, and let $M : X \to 2^X$ be relative maximal monotone. Then the following statements are mutually equivalent:*
(i) *An element $u \in X$ is a solution to* (1).
(ii) *For an $u \in X,$ we have*

$$u = R_{\rho,A}^M(u),$$

where

$$R_{\rho,A}^M(u) = (I + \rho M)^{-1}(u).$$

Proof. It follows from the definition of resolvent operator corresponding to M. ■

Theorem 3.2. *Let X be a real Hilbert space, let $A : X \to X$ be $(r) -$ strongly monotone, and let $M : X \to 2^X$ be relative maximal monotone. Furthermore, suppose that $AoR_{\rho_k,A}^M$ is cocoercive with respect to $R_{\rho_k,A}^M$.*
(i) *For an arbitrarily chosen initial point $x^0,$ suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm*

$$x^{k+1} \approx R_{\rho_k,A}^M(x^k)$$

such that

$$\|x^{k+1} - R_{\rho_k,A}^M(x^k)\| \leq \epsilon_k, \tag{9}$$

where $\Sigma_{k=0}^\infty \epsilon_k < \infty,$ $R_{\rho_k,A}^M = (I + \rho_k M)^{-1},$ $r > 1,$ and the scalar sequence $\{\rho_k\}$ satisfies $\rho_k \uparrow \rho \leq \infty.$
Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1).

(ii) *In addition to assumptions in (i), we further suppose that, for an arbitrarily chosen initial point $x^0,$ the sequence $\{x^k\}$ is generated by the proximal point algorithm*

$$x^{k+1} \approx R_{\rho_k,A}^M(x^k)$$

such that

$$\|x^{k+1} - R^M_{\rho_k,A}(x^k)\| \leq \delta_k \|x^{k+1} - x^k\|, \tag{10}$$

where $\delta_k \to 0,$ $R^M_{\rho_k,A} = (I + \rho_k M)^{-1}$, *and the scalar sequences* $\{\delta_k\}$ *and* $\{\rho_k\}$, *respectively, satisfy* $\Sigma_{k=0}^{\infty}\delta_k < \infty,$ *and* $\rho_k \uparrow \rho \leq \infty.$
Then the following implications hold:
(iii) *The sequence* $\{x^k\}$ *converges strongly to a solution of* (1).
(iv) *Rate of convergence*

$$0 \leq lim_{k\to\infty} \frac{\delta_k + (r)^{-1}}{1 - \delta_k} < 1,$$

where $\frac{1}{r} < 1.$

Proof. Suppose that x^* is a zero of $M.$ We start with the proof for

$$\|R^M_{\rho_k,A}(x^k) - R^M_{\rho_k,A}(x^*)\| \leq \frac{1}{r}\|x^k - x^*\|.$$

It follows from the definition of the generalized resolvent operator $R^M_{\rho_k,A}$ and the relative monotonicity of M with respect to A that

$$\rho^{-1}\langle x^k - x^* - (R^M_{\rho_k,A}(x^k) - R^M_{\rho_k,A}(x^*)), A(R^M_{\rho_k,A}(x^k)) - A(R^M_{\rho_k,A}(x^*))\rangle \geq 0.$$

It further follows that

$$\begin{aligned} &\langle x^k - x^*, A(R^M_{\rho_k,A}(x^k)) - A(R^M_{\rho_k,A}(x^*))\rangle \\ \geq &\langle R^M_{\rho_k,A}(x^k) - R^M_{\rho_k,A}(x^*), A(R^M_{\rho_k,A}(x^k)) - A(R^M_{\rho_k,A}(x^*))\rangle. \end{aligned}$$

Since A is (r) – strongly monotone (and hence (r) – expanding) and $AoR^M_{\rho_k,A}$ is cocoercive with respect to $R^M_{\rho_k,A},$ the inequality follows.
Next, we move to estimate

$$\begin{aligned} \|x^{k+1} - x^*\| &\leq \|R^M_{\rho_k,A}(x^k) - x^*\| + \epsilon_k \\ &= \|R^M_{\rho_k,A}(x^k) - R^M_{\rho_k,A}(x^*)\| + \epsilon_k \\ &\leq \frac{1}{r}\|x^k - x^*\| + \epsilon_k. \end{aligned}$$

Since $r > 1,$ combining for all k, we have

$$\begin{aligned} &\|x^{k+1} - x^*\| \leq \|x^0 - x^*\| + \sum_{i=0}^{k} \epsilon_i \ \forall\ i \\ &\leq \|x^{k+1} - x^*\| \leq \|x^0 - x^*\| + \sum_{k=0}^{\infty} \epsilon_k. \end{aligned} \tag{11}$$

Hence, $\{x^k\}$ is bounded.

Next we turn our attention to the main proof. Since

$$\|x^{k+1}-x^*\| \leq \|x^{k+1}-R^M_{\rho_k,A}(x^k)\| + \|R^M_{\rho_k,A}(x^k)-R^M_{\rho_k,A}(x^*)\|, \tag{12}$$

and

$$\|x^{k+1}-R^M_{\rho_k,A}(x^k)\| \leq \delta_k\|x^{k+1}-x^k\| \leq \delta_k[\|x^{k+1}-x^*\|+\|x^k-x^*\|], \tag{13}$$

we have

$$\begin{aligned}\|x^{k+1}-x^*\| &\leq \|x^{k+1}-R^M_{\rho_k,A}(x^k)\| \\ &+ \|R^M_{\rho_k}(x^k)-R^M_{\rho_k,A}(x^*)\| \\ &\leq \delta_k[\|x^{k+1}-x^*\|+\|x^k-x^*\|] \\ &+ \frac{1}{r}\|x^k-x^*\| \, for \, k \geq k',\end{aligned}$$

where $\frac{1}{r} < 1$.
It follows that

$$\|x^{k+1}-x^*\| \leq \frac{(r)^{-1}+\delta_k}{1-\delta_k}\|x^k-x^*\| \, for \ k \ \geq k'. \tag{14}$$

It looks favorably that (14) holds since $\frac{1}{r} < 1$ (seems to hold) and $\delta_k \to 0$.
Hence, the sequence $\{x^k\}$ converges strongly to x^*.
Finally, to show the uniqueness of the solution to (1), assume that x^* is a zero of $M.$ Then using $\|x^k-x^*\| \leq \|x^0-x^*\|+\sum_{k=0}^{\infty}\epsilon_k \, \forall\, k,$ we have

$$a^* = lim_{k\to\infty} inf\|x^k-x^*\|$$

is nonnegative and finite, and as a result, $\|x^k-x^*\| \to a^*.$ Consider x_1^* and x_2^* as two limit points of $\{x^k\}.$ Then we have

$$\|x^k-x_1^*\| = a_1, \ \|x^k-x_2^*\| = a_2$$

and both exist and are finite. If we express

$$\|x^k-x_2^*\|^2 = \|x^k-x_1^*\|^2 + 2\langle x^k-x_1^*, x_1^*-x_2^*\rangle + \|x_1^*-x_2^*,$$

then it follows that

$$lim_{k\to\infty}\langle x^k-x_1^*, x_1^*-x_2^*\rangle = \frac{1}{2}[a_2^2-a_1^2-\|x_1^*-x_2^*\|^2].$$

Since x_1^* is a limit point of $\{x^k\},$ the left hand side limit must tend to zero. Therefore,

$$a_2^2 = a_1^2 - \|x_1^*-x_2^*\|^2.$$

Similarly, we obtain

$$a_1^2 = a_2^2 - \|x_1^*-x_2^*\|^2.$$

This results in $x_1^* = x_2^*.$ ■

4. Concluding Remark

It seems that the notion of relative maximal monotone mappings is significantly application-enhanced in a way that it can be applied to generalize the Yosida approximation to the context of first-order nonlinear evolution equations as well as evolution inclusions. We define the generalized Yosida Approximation M_ρ by

$$M_\rho = \rho^{-1}(I - (I + \rho M)^{-1}) \, for \, \rho > 0, \tag{15}$$

where $(I + \rho M)^{-1}$ is the resolvent operator corresponding to $M : X \to 2^X$.

References

[1] R. P. Agarwal, and R. U. Verma, Role of relative A –maximal monotonicity in overrelaxed proximal point algorithms with applications, *Journal of Optimization Theory and Applications* **143** (1), (2009), 1–15.

[2] R. P. Agarwal, and R. U. Verma, The over-relaxed η –proximal point algorithm and nonlinear variational inclusion problems, *Nonlinear Functional Analysis and Applications,* in press.

[3] H. Y. Lan, Y. J. Cho, and R. U. Verma, Nonlinear relaxed cocoercive variational inclusions involving (A,η) –accretive mappings in Banach spaces, *Computers & Mathematics with Applications* **51**, (2006), 1529–1538.

[4] A. Moudafi, and M. Thera, Finding a zero of the sum of two maximal monotone operators, *Journal of Optimization Theory and Applications* **94**, (1997), 425–448.

[5] R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM Journal of Control and Optimization* **14**, (1976), 877–898.

[6] R. T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, *Mathematics of Operations Research* **1**, (1976), 97–116.

[7] R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics* **33**, (1970), 209–216.

[8] P. Tossings, The perturbed proximal point algorithm and some of its applications , *Applied Mathematics and Optimization* **29**, (1994), 125–159.

[9] R. U. Verma, A – monotonicity and its role in nonlinear variational inclusions, *Journal of Optimization Theory and Applications* **129**(3), (2006), 457–467.

[10] R. U. Verma, A –monotone nonlinear relaxed cocoercive variational inclusions, *Central European Journal of Mathematics* **5**(2), (2007), 386–396.

[11] R. U. Verma, A generalization to variational convergence for operators, *Advances in Nonlinear Variational Inequalities* **11**(2), (2008), 97–101.

[12] R. U. Verma, Approximation solvability of a class of nonlinear set-valued inclusions involving (A,η) – monotone mappings, *Journal of Mathematical Analysis and Applications* **337**, (2008), 969–975.

[13] E. Zeidler, *Nonlinear Functional Analysis and its Applications* I, Springer-Verlag, New York, 1986.

[14] E. Zeidler, *Nonlinear Functional Analysis and its Applications* II/A, Springer-Verlag, New York, 1990.

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications* III, Springer-Verlag, New York, 1985.