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Stabilized Milstein Type Methods for Stiff Stochastic Systems

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Abstract. *In this paper we discuss Milstein type methods with implicitness for solving Itô stochastic differential equations (SDEs). For different Milstein type methods, the regions of mean-square (MS) stability are examined. The drift implicit balanced Milstein (DIBM) method and the semi-implicit balanced Milstein (SIBM) method are proposed in this paper. The obtained results show that the MS-stability of Milstein type methods with implicitness is better than that of the classical Milstein method. This is also verified by some numerical examples.*

Key words: Stochastic differential equations, Milstein method, Implicitness, MS-stability, Stiff equations.

AMS Subject Classifications : 65C20

1. Introduction

The importance of numerical methods for stochastic differential equations (SDEs) cannot be overemphasized as SDEs are used in the modeling of many biological, chemical, physical, and economical systems. In this paper we consider numerical methods for the strong solution of Itŏ SDEs

$$dy(t) = f(t,y(t))dt + g(t,y(t))dW(t), \quad y(t_0) = y_0, \tag{1}$$

where $y(t)$ is a random variable with value in $\mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is called the drift function, $g : \mathbb{R}^m \to \mathbb{R}^m$ is called the diffusion function, and $W(t)$ is a one-dimensional Wiener process, whose increment $\Delta W(t) = W(t+\Delta t) - W(t)$ is a Gaussian random variable $N(0,\Delta t)$. For simplicity in this paper numerical methods on a given time interval $[t_0, T]$ are fixed by schemes based on equidistant time discretization points $t_n = t_0 + nh$, $n = 0, 1, \cdots, N$ with step size $h = (T - t_0)/N$, $N = 1, 2, \cdots\cdots$. We recall the concepts of accuracy for numerical integration of

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SDEs. A method is said to have a strong order p (respectively, weak order of v) if there exists a constant C such that

$$E(|y(\tau) - y_N|) \leqslant Ch^p \text{ (strong)}, \quad |E(\varphi(y_N)) - E(\varphi(y(\tau)))| \leqslant Ch^v \text{ (weak)}$$

for any fixed $\tau = t_0 + Nh \in [t_0, T]$ and for all functions $\varphi : \mathbb{R}^m \to \mathbb{R}$, $2(v+1)$ times continuously differentiable and for which all partial derivatives have polynomial growth. In this paper, we focus our attention on schemes that converge in the strong sense. For SDE (1), the well-known Euler-Maruyama method with strong order 0.5 is given by

$$y_{n+1} = y_n + f(t_n, y_n)h + g(t_n, y_n)\Delta W_n, \tag{2}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$, $n = 1, 2, \cdots, N-1$ and $y_0 = y(t_0)$, see [20, 4, 26]. By including from the Itô-Taylor expansion the additional term

$$g(t_n, y_n)\frac{\partial g}{\partial y}(t_n, y_n)\int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s = \frac{1}{2}g(t_n, y_n)\frac{\partial g}{\partial y}(t_n, y_n)[(\Delta W_n)^2 - h],$$

Milstein has presented in [21] an important (Milstein) method with strong order 1.0, namely

$$y_{n+1} = y_n + f(t_n, y_n)h + g(t_n, y_n)\Delta W_n + \frac{1}{2}g(t_n, y_n)\frac{\partial g}{\partial y}(t_n, y_n)[(\Delta W_n)^2 - h]. \tag{3}$$

By the multiplicative ergodic theorem of Osceledec (see [3]), SDE (1) is said to be stiff if its linearized system, given by

$$dX(t) = AX(t)dt + BX(t)dW(t), \quad X(t_0) = x_0,$$

has m negative Lyapunov exponents, $\lambda_m \leqslant \lambda_{m-1} \leqslant, \cdots, \lambda_1$ given by

$$\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t}\ln|X(t, x_0)|,$$

such that $\lambda_m \ll \lambda_1$. Stochastic stiffness is a generalization of the deterministic notion of stiffness, so a stiff ordinary differential equation is also stiff in the stochastic sense [18, 2].

In recent years many efficient numerical methods are constructed for solving different types of SDEs with different properties (for example, see [24, 19, 23, 12]). In particular, several authors have presented different efficient implicit methods for stiff SDEs (see [22, 8, 9, 30, 27]). In addition, several authors have found some explicit methods with better stability for SDEs (see [1, 29]). In designing efficient numerical methods for SDEs, an important criterion is that of stability. The Mean-square (MS) stability is a stochastic version of absolute stability, and it is a very important concept in numerical simulation of SDEs. A suitable way to find numerical schemes for stiff SDEs is analysis of MS-stability. In this paper we discuss MS-stability of Milstein type methods with implicitness for solving Itŏ SDEs. In Section 2, we study MS-stability of four methods which are derived from general Milstein method. The MS-stability of balanced Milstein methods is considered in Section 3. Numerical results are reported in Section 4.

2. Mean Square Stability of Milstein Type Methods

We begin this section with the definition of mean-square stability. Other notions of stability can be found, for example, in [5, 14, 13, 6].

Definition 2.1. The equilibrium position, $y(t) \equiv 0$, is said to be mean-square stable if for every $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\| \, y(t) \, \| < \varepsilon, \quad for \;\; all \;\; t \geqslant 0 \;\; and \;\; |y_0| < \delta_1, \tag{4}$$

where $\| \, y(t) \, \| = (E|y(t)|^2)^{1/2}$. If, in addition to (4), there exists a $\delta_2 > 0$ such that

$$\lim_{t\to\infty} \|y(t)\| = 0, \quad for \;\; all \;\; |y_0| < \delta_2,$$

then the equilibrium position is said to be asymptotically mean square stable.

Definition 2.2. Suppose that the equilibrium position of Itô's SDE (1) is asymptotically mean-square stable. Then a numerical scheme that produces the iterations y_n to approximate the solution $y(t)$ of (1) is said to be asymptotically mean-square stable if

$$\lim_{n\to\infty} \| \, y_n \, \| = 0.$$

We apply one-step scheme to the scalar linear test equation

$$dy(t) = ay(t)dt + by(t)dW(t), \quad y(t_0) = y_0, \tag{5}$$

with known solution $y(t) = y_0 e^{(a-b^2/2)\,t + b\,W(t)}$, which is represented by

$$y_{n+1} = R(a,b,h,J)y_n,$$

where J is the standard Gaussian random variable $J = \Delta W_n/\sqrt{h} \sim N(0,1)$. Saito and Mitsui [25] introduced the following definition of mean-square (MS) stability.

Definition [25] **2.3**. The numerical method is said to be MS-stable for a, b, h if

$$\overline{R}(a,b,h) = E(R^2(a,b,h,J)) < 1.$$

$\overline{R}(a,b,h)$ is called MS-stability function of the numerical method.

The Euler-Maruyama method (2) is an explicit method. In fact, there is no simple stochastic counterpart of the deterministic implicit Euler method, i.e., the method

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + g(t_{n+1}, y_{n+1})\Delta W_n$$

fails because, for example, we have $E|(1 - ah - b\Delta W_n)^{-1}| = +\infty$ for linear SDE (5). Nevertheless, a way to introduce implicitness in the numerical treatment could be to look at a higher order explicit strong method and try to introduce implicitness there. For Milstein method (3), by introducing implicitness in $f(t_n, y_n)h$, we arrive at a drift implicit Milstein (DIM) method

$$\begin{aligned} y_{n+1} &= y_n + f(t_{n+1}, y_{n+1})h + g(t_n, y_n)\Delta W_n \\ &\quad + \tfrac{1}{2} g(t_n, y_n) \tfrac{\partial g}{\partial y}(t_n, y_n)[(\Delta W_n)^2 - h]. \end{aligned} \tag{6}$$

Applying the DIM method (6) to the linear test equation (5), we obtain

$$y_{n+1} = R_1(p,q,J)y_n,$$

where $p = ah$, $q = b\sqrt{h}$ and

$$R_1(p,q,J) = \frac{1 + qJ + \frac{1}{2}q^2J^2 - \frac{1}{2}q^2}{1 - p}.$$

The MS-stability function of the DIM method is given by

$$\overline{R_1}(p,q) = \frac{1+q^2+\frac{1}{2}q^4}{(1-p)^2}.$$

The DIM method will be MS-stable if $\overline{R_1}(p,q) < 1$.
In addition, we can analyze the term

$$\frac{1}{2}g(t_n,y_n)\frac{\partial g}{\partial y}(t_n,y_n)[(\Delta W_n)^2-h] = \frac{1}{2}g(t_n,y_n)\frac{\partial g}{\partial y}(t_n,y_n)(\Delta W_n)^2 - \frac{1}{2}g(t_n,y_n)\frac{\partial g}{\partial y}(t_n,y_n)h$$

and introduce partial implicitness. This leads to a semi-implicit Milstein (SIM) method [22], namely to

$$\begin{aligned} y_{n+1} &= y_n + [f(t_{n+1},y_{n+1}) - \tfrac{1}{2}g(t_{n+1},y_{n+1})\tfrac{\partial g}{\partial y}(t_{n+1},y_{n+1})]h \\ &\quad + \tfrac{1}{2}g(t_n,y_n)\tfrac{\partial g}{\partial y}(t_n,y_n)(\Delta W_n)^2. \end{aligned} \tag{7}$$

Applying the SIM method (7) to the linear test equation (6), we obtain

$$y_{n+1} = R_2(p,q,J)\, y_n\,,$$

where

$$R_2(p,q,J) = \frac{1+qJ+\frac{1}{2}q^2J^2}{1-p+\frac{1}{2}q^2}.$$

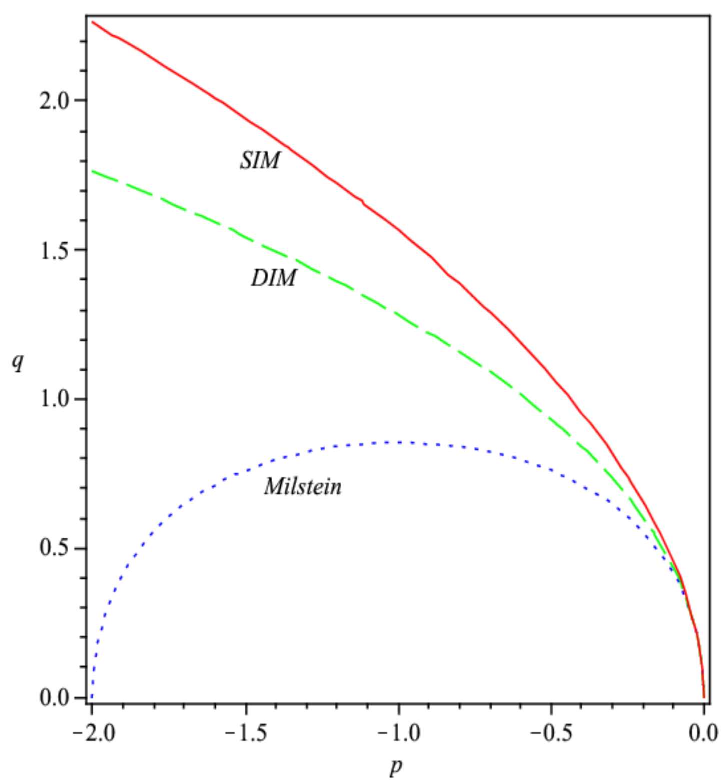

Figure1: MS-stable regions of Milstein type methods

The MS-stability function of the SIM method is given by

$$\overline{R_2}(p,q) = \frac{1+2q^2+\frac{3}{4}q^4}{(1-p+\frac{1}{2}q^2)^2}.$$

The modified split-step backward Milstein (MSSBM) method will be MS-stable if $\overline{R_2}(p,q) < 1$.

Figure 1 gives the MS-stable regions of the Milstein, DIM and SIM methods. The MS-stable regions are the areas under the plotted curves and symmetric about the p-axis. The MS-stability property of the DIM method is better than that of Milstein method, and the MS-stability property of the SIM method is better than that of the DIM method. The MS-stable regions of DIM and SIM methods are semi-infinite.

Let us rewrite the SDE (1) in the following form:

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad y(t_0) = y_0, t \in [t_0, T], y \in \mathbb{R}^m. \tag{8}$$

For SDE (8), Higham, Mao and Stuart [15] presented a split-step backward Euler method, namely

$$\begin{aligned} \overline{Y}_n &= y_n + hf(\overline{Y}_n), \\ y_{n+1} &= \overline{Y}_n + \Delta W_n g(\overline{Y}_n). \end{aligned} \tag{9}$$

Furthermore, they proved the convergence of method (9) under the one-sided Lipschitz condition.

Using the same implicit splitting technique, Wang [28] presented the drifting split-step backward Milstein (DSSBM) method, given by

$$\begin{aligned} \overline{Y}_n &= y_n + hf(\overline{Y}_n), \\ y_{n+1} &= \overline{Y}_n + \Delta W_n g(\overline{Y}_n) + \tfrac{1}{2}g(\overline{Y}_n)g'(\overline{Y}_n)[(\Delta W_n)^2 - h]. \end{aligned} \tag{10}$$

In addition, using the fully splitting technique for deterministic terms, Wang [28] obtained the following modified split-step backward Milstein (MSSBM) method, namely

$$\begin{aligned} \overline{Y}_n &= y_n + h[f(\overline{Y}_n) - \tfrac{1}{2}g(\overline{Y}_n)g'(\overline{Y}_n)], \\ y_{n+1} &= \overline{Y}_n + \Delta W_n g(\overline{Y}_n) + \tfrac{1}{2}g(\overline{Y}_n)g'(\overline{Y}_n)(\Delta W_n)^2. \end{aligned} \tag{11}$$

The convergence of methods (10) and (11) is proved in [28]. Applying the DSSBM method (10) and the MSSBM method (11) to linear test equation (5), we can obtain that the MS-stability function and the MS-stable region of the DSSBM method are the same as those of the DIM method, and the same is true for the MSSBM and SIM methods.

3. Mean Square Stability of Balanced Milstein Methods

Milstein, Platen and Schurz [22] presented the class of balanced implicit (BI) methods for stiff SDEs

$$y_{n+1} = y_n + f(t_n, y_n)h + g(t_n, y_n)\Delta W_n + C_n(y_n - y_{n+1}), \tag{12}$$

where

$$C_n = c_0(t_n,y_n)h + c_1(t_n,y_n)|\Delta W_n|.$$

In this method the functions c_0 and c_1 are called control functions. The control functions must be bounded and have to satisfy the inequality

$$1 + c_0(t_n,y_n)h + c_1(t_n,y_n)|\Delta W_n| > 0.$$

Using the idea of the BI method and combining it with the Milstein method, Kahl [16] presented a balanced Milstein (BM) method (see also Kahl and Schurz [17]), namely

$$\begin{aligned} y_{n+1} &= y_n + f(t_n,y_n)h + g(t_n,y_n)\Delta W_n \\ &\quad + \tfrac{1}{2}g(t_n,y_n)\tfrac{\partial g}{\partial y}(t_n,y_n)[(\Delta W_n)^2 - h] + C_n(y_n - y_{n+1}), \end{aligned} \tag{13}$$

where

$$C_n = c_0(t_n,y_n)h + c_2(t_n,y_n)[(\Delta W_n)^2 - h], \tag{14}$$

is restricted by

$$1 + c_0(t_n,y_n)h + c_2(t_n,y_n)[(\Delta W_n)^2 - h] > 0. \tag{15}$$

Applying the BM method (13) – (14) with $c_0 = -a$ and $c_2 = 0$ to the linear test equation (5), we obtain that the MS-stability function and the MS-stable region of the BM method are the same as for the DIM method.
For the BM method (13) – (14), introducing implicitness in $f(y_n)h$, leads to the drift implicit Milstein (DIBM) method, namely to

$$\begin{aligned} y_{n+1} &= y_n + f(t_{n+1},y_{n+1})h + g(t_n,y_n)\Delta W_n \\ &\quad + \tfrac{1}{2}g(t_n,y_n)\tfrac{\partial g}{\partial y}(t_n,y_n)[(\Delta W_n)^2 - h] + C_n(y_n - y_{n+1}), \end{aligned} \tag{16}$$

where

$$C_n = c_0(t_n,y_n)h + c_2(t_n,y_n)[(\Delta W_n)^2 - h]. \tag{17}$$

The control functions c_0, c_2 satisfy the inequality (15). Applying the DIBM method (16) – (17) with $c_0 = -a$ and $c_2 = 0$ to the linear test equation (5), we can obtain

$$y_{n+1} = R_3(p,q,J)y_n,$$

where

$$R_3(p,q,J) = \frac{1 + p + qJ + \frac{1}{2}q^2J^2 - \frac{1}{2}q^2}{1 - 2p}.$$

The MS-stability function of the DIBM method is given by

$$\overline{R_3}(p,q) = \frac{1 + 2p + p^2 + q^2 + \frac{1}{2}q^4}{(1 - 2p)^2}.$$

The DIBM method will be MS-stable if $\overline{R_3}(p,q) < 1$.

Similar to the SIM method (7), introducing partial implicitness in the BM method (13) – (14), we obtain the semi-implicit balanced Milstein (SIBM) method

$$\begin{aligned} y_{n+1} &= y_n + [f(t_{n+1},y_{n+1}) - \tfrac{1}{2}g(t_{n+1},y_{n+1})\tfrac{\partial g}{\partial y}(t_{n+1},y_{n+1})]h \\ &\quad + \tfrac{1}{2}g(t_n,y_n)\tfrac{\partial g}{\partial y}(t_n,y_n)(\Delta W_n)^2 + C_n(y_n - y_{n+1}), \end{aligned} \tag{18}$$

where

$$C_n = c_0(t_n,y_n)h + c_2(t_n,y_n)[(\Delta W_n)^2 - h]. \tag{19}$$

The control functions c_0, c_2 also satisfy the inequality (15). Applying the SIBM method (18) – (19) with $c_0 = -a$ and $c_2 = 0$ to the linear test equation (5), we obtain

$$y_{n+1} = R_4(p,q,J)y_n,$$

where

$$R_4(p,q,J) = \frac{1 + p + qJ + \frac{1}{2}q^2J^2}{1 - 2p + \frac{1}{2}q^2}.$$

The MS-stability function of SIBM method is given by

$$\overline{R_4}(p,q) = \frac{1 + 2p + p^2 + 2q^2 + pq^2 + \frac{3}{4}q^4}{(1 - 2p + \frac{1}{2}q^2)^2}.$$

The SIBM method will be MS-stable if $\overline{R_4}(p,q) < 1$.

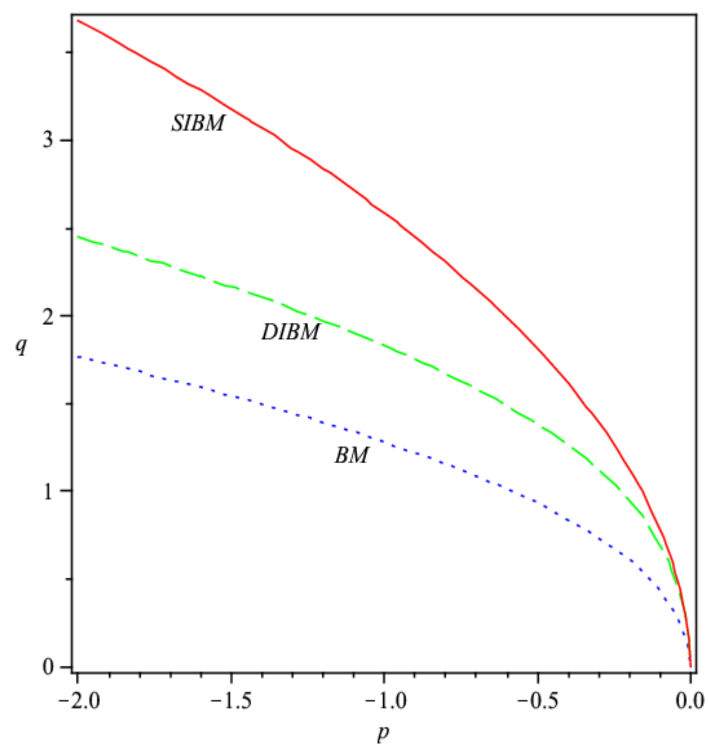

Figure 2: MS-stable regions of balanced Milstein methods

Figure 2 illustrates the MS-stable regions of the BM, DIBM and SIBM methods. The MS-stabe regions are the areas under the plotted curves and symmetric about the p-axis. The SMS-stability property of the DIBM method is better than that of the BM method. So is the MS-stability property of the SIBM when compared with that of the DIBM method. The

MS-stability regions of the BM, DM and SIM methods are semi-infinite.

4. Numerical Results

Numerical results are reported in this section to confirm the convergence properties and stability properties of several Milstein type methods. Denoting y_{iN} as the numerical approximation to $y_i(t_N)$ at step point t_N in the i-th simulation of all 5000 simulations, we use means of absolute errors M, strong order 1.0 convergence rates $R_{1.0}$, defined by

$$M = \frac{1}{5000}\sum_{i=1}^{5000}|y_{iN} - y_i(t_N)|, \quad R_{1.0} = \frac{M}{h},$$

to measure the accuracy and convergence property of Milstein type methods.

The test equation is a 2-dimensional linear SDE system whose Itŏ form is given by

$$dy(t) = Uy(t)dt + Vy(t)dW(t), \qquad y(t_0) = y_0, t \in [0,\ 1], y \in \mathbb{R}^2, \tag{20}$$

where U and V are the matrices

$$U = \begin{pmatrix} -u & u \\ u & -u \end{pmatrix}, \quad V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}. \tag{21}$$

Table 1 : Errors and convergence rate for (20) and (21) ($u = 5, v = 10$)

	Milstein		SIM		SIBM	
h	M	$R_{1.0}$	M	$R_{1.0}$	M	$R_{1.0}$
2^{-1}	$8.30c-3$	$1.66c-2$	$1.20c-3$	$2.40c-3$	$1.10c-3$	$2.20c-3$
2^{-2}	$5.90c-3$	$2.36c-2$	$1.30c-3$	$5.20c-3$	$1.20c-3$	$4.80c-3$
2^{-3}	$4.30c-3$	$3.44c-2$	$1.30c-3$	$1.04c-2$	$1.20c-3$	$9.60c-3$
2^{-4}	$2.80c-3$	$4.48c-2$	$9.73c-4$	$1.56c-2$	$8.71c-4$	$1.39c-2$
2^{-5}	$1.40c-3$	$4.48c-2$	$4.16c-4$	$1.33c-2$	$3.30c-4$	$1.05c-2$
2^{-6}	$6.05c-4$	$3.87c-2$	$9.25c-5$	$5.92c-3$	$4.99c-5$	$3.19c-3$

The exact solution of this equation is given by [19]

$$y(t) = P\begin{pmatrix} \exp(\rho^+(t)) & 0 \\ 0 & \exp(\rho^-(t)) \end{pmatrix}P^{-1}y_0, \quad P = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where $\rho^{\pm}(t) = (-u - ½\, v^2 \pm u)t + vW(t)$ and $P^{-1} = P$.

This equation is stiff in the deterministic (or stochastic) component if u (or v) is large. The stiffness of this linear system increases quadratically in terms of v. Suitable numerical results can be obtained only with smaller stepsize if this stochastic system is stiff (see [27,7]). Table 1 gives the errors and strong convergence rates of Milstein, SIM and SIBM methods when solving (20) and (21) with $u = 5$, $v = 10$ and $y_0 = (1, 2)^T$. The accuracy of the SIM method is better than that of Milstein method. Clearly, the accuracy of the SIBM method is better than that of the SIM method.

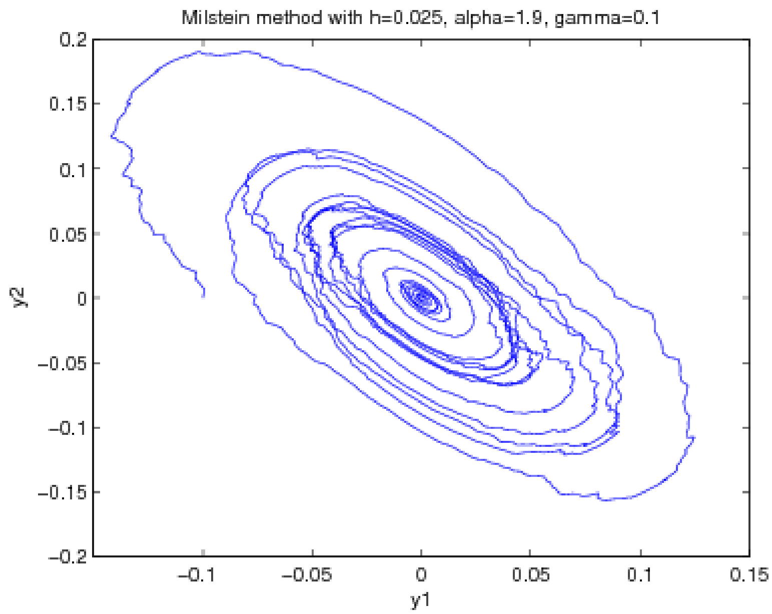

Figure 3: Numerical simulation of the system (22) by Milstein method

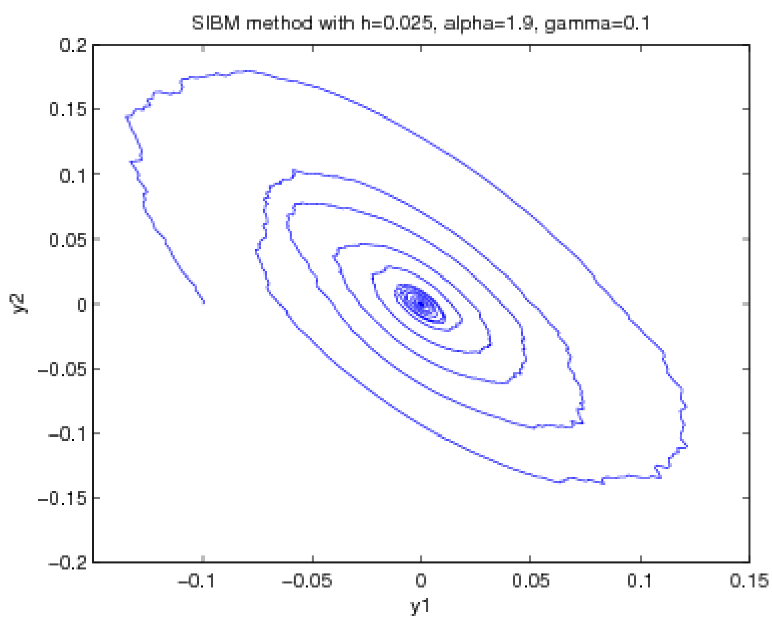

Figure 4: Numerical simulation of the system (22) by SIBM method

The second test equation is a stochastic version of the Brusselator system (see [10,11,19])

$$\begin{aligned} dy_1(t) &= ((\alpha-1)y_1(t)+\alpha y_1^2(t)+(y_1(t)+1)^2 y_2(t))dt+\gamma y_1(t)(1+y_1(t))\, dW(t), \\ dy_2(t) &= (-\alpha y_1(t)-\alpha y_1^2(t)-(y_1(t)+1)^2 y_2(t))dt-\gamma y_1(t)(1+y_1(t))\, dW(t), \end{aligned} \tag{22}$$

which model unforced periodic oscillations in certain chemical reactions

Using Milstein and SIBM methods with $h = 0.025$, in Figure 3 and in Figure 4, respectively, we give the numerical simulation of equation (22) with $\alpha = 1.9$, $\gamma = 0.1$, $0 \le t \le 125$ starting at $(y_1(0), y_2(0)) = (-0.1, 0)$. We observe in Figure 4 for the semi-implicit balanced Milstein method the approximate trajectories stay close to the origin $(0,0)$, which replicates the behavior of the exact solution. Here, the semi-implicit balanced Milstein method yields a better approximation than the Milstein method where the approximation is more damped and thus approaches the origin too fast.

5. Conclusions

In this paper we have analyzed mean-square stability of Milstein type methods with implicitness for solving Itŏ SDEs. The drift implicit balanced Milstein (DIBM) method and the semi-implicit balanced Milstein (SIBM) method are proposed in this paper. By comparing the MS-stability of these different methods, we have demonstrated that the semi-implicit balanced Milstein method is more suitable for stiff SDEs. This is also verified by a numerical example. We will consider constructing methods with better stability properties in future work.

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