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On the Boundedness Character of a Rational System

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Abstract. *We investigate the boundedness character of solutions of a rational system with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.*

Key words: Boundedness, Rational systems.

AMS Subject Classifications : 39A10.

1. Introduction

We investigate the boundedness character of solutions of the rational system in the plane

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1+\beta_1 x_n}{x_n+y_n} \\ \\ y_{n+1} &= \frac{\alpha_2+\beta_2 x_n+y_n}{A_2+x_n} \end{aligned}\right\}, \quad n = 0,1,\ldots \tag{1}$$

with $A_2 > 0$ and the remaining parameters nonnegative, and with arbitrary nonnegative initial conditions such that the denominators are always positive.
In the numbering system which was introduced in [6], System (1) contains the following 12 special cases of rational systems:

$$(12,14),\quad (12,29),\quad (12,35),\quad (12,47),$$
$$(14,30),\quad (15,14),\quad (15,29),\quad (15,35),$$
$$(15,47),\quad (29,30),\quad (30,35),\quad (30,47).$$

The boundedness of the six special cases:

$$(12,47),\quad (15,14),\quad (15,29),\quad (15,35),\quad (15,47),\quad \text{and}\quad (30,47)$$

was established in [5]. Here we establish the boundedness character of solutions in the

remaining six cases. We strongly believe that the methods and techniques we develop here to understand the boundedness of various special cases of rational systems will also be useful in analyzing the boundedness character of solutions in any mathematical model that involves systems of difference equations. For some work on rational systems see [5]-[6], [9]-[13], and [23]. Also for some basic results in the area of difference equations and systems see [1]-[4], [7]-[8], [14]-[22] and [24].
The following lemma will be useful in the sequel.

Lemma 1.1. *Let $\{y_n\}$ be an arbitrary sequence of positive numbers and let $\{x_n\}$ be a solution of the difference equation*

$$x_{n+1} = \frac{\alpha_1}{x_n + y_n}, \quad n = 0, 1, \ldots .$$

Then

$$x_{n+1} - x_{n-1} = \frac{-y_n\left(x_{n-1}^2 + y_{n-1}x_{n-1} - \alpha_1 \cdot \frac{y_{n-1}}{y_n}\right)}{\alpha_1 + y_n(x_{n-1} + y_{n-1})}.$$

2. The Boundedness Character of System (12,14)

In this section we investigate the boundedness character of solutions of the system:

$$(12,14): \quad \left.\begin{array}{rcl} x_{n+1} & = & \frac{\alpha_1}{x_n+y_n} \\ \\ y_{n+1} & = & \frac{\gamma_2 y_n}{A_2+x_n} \end{array}\right\}, \quad n = 0, 1, \ldots \tag{2}$$

with positive parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.

We establish that for every solution $\{x_n, y_n\}$, the first component $\{x_n\}$ is always bounded, for all values of the parameters and for all initial conditions. For the second component, $\{y_n\}$, we show that it is unbounded in a certain region of the parameters and for some initial conditions.

Theorem 2.1. *Let $\{x_n, y_n\}$ be a solution of System* (2). *Then the sequence $\{x_n\}$ is bounded.*

Proof. Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \text{ and } x_{n_i+1} > x_n, \quad \text{for} \quad n < n_i + 1. \tag{3}$$

Then, clearly

$$x_{n_i}, y_{n_i}, y_{n_i-1} \to 0 \text{ and } x_{n_i-1} \to \infty.$$

In view of Lemma 1.1,

$$x_{n_i+1} - x_{n_i-1} = \frac{-y_{n_i}\left(x_{n_i-1}^2 + y_{n_i-1}x_{n_i-1} - \alpha_1 \cdot \frac{y_{n_i-1}}{y_{n_i}}\right)}{\alpha_1 + y_{n_i}(x_{n_i-1} + y_{n_i-1})} \tag{4}$$

and

$$\frac{y_{n_i-1}}{y_{n_i}} = \frac{A_2}{\alpha_2} + \frac{x_{n_i-1}}{\alpha_2},$$

it follows that

$$x_{n_i-1}^2 + y_{n_i-1}x_{n_i-1} - \alpha_1 \cdot \frac{y_{n_i-1}}{y_{n_i}} = x_{n_i-1}^2 + (y_{n_i-1} - \frac{\alpha_1}{\alpha_2})x_{n_i-1} - \frac{\alpha_1 A_2}{\alpha_2} > 0$$

and so

$$x_{n_i+1} < x_{n_i-1},$$

which is a contradiction. ■

Theorem 2.2. *Assume that*

$$\gamma_2 > A_2.$$

Let $\{x_n, y_n\}$ *be a solution of System* (2) *with initial conditions*

$$x_0 < \gamma_2 - A_2 \text{ and } y_0 > \max\left\{\frac{\alpha_1}{\gamma_2 - A_2}, \frac{\alpha_1 - (\gamma_2 - A_2)^2}{\gamma_2 - A_2}\right\}.$$

Then

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty.$$

Proof. Indeed,

$$x_1 = \frac{\alpha_1}{x_0 + y_0} < \frac{\alpha_1}{y_0} < \gamma_2 - A_2 \quad \text{and} \quad y_1 = \frac{\gamma_2}{A_2 + x_0} \cdot y_0 > y_0$$

and by induction, we see that

$$x_{n+1} < \gamma_2 - A_2 \quad \text{and} \quad y_{n+1} > y_n, \quad \text{for} \quad n \geq 0$$

from which the result follows. ■

The following result describes the global character of solutions of System (2) when

$$\gamma_2 \leq A_2. \tag{5}$$

Theorem 2.3. *Assume that* (5) *holds. Then every solution* $\{x_n, y_n\}$ *of system* (2) *is bounded and converges to a* (*not necessarily prime*) *period – two solution*.

Proof. The proof is straightforward and will be omitted. ■

3. The Boundedness Character of System (12,29)

In this section we investigate the boundedness character of solutions of the system:

$$(12,29):\quad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1}{x_n+y_n} \\ y_{n+1} &= \frac{\alpha_2+y_n}{A_2+x_n} \end{aligned}\right\},\quad n=0,1,\ldots \tag{6}$$

with positive parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.
We establish that for every solution $\{x_n, y_n\}$, the first component $\{x_n\}$ is always bounded, for all values of the parameters and for all initial conditions. For the second component, $\{y_n\}$, we show that it is unbounded in a certain region of the parameters and for some initial conditions.

Theorem 3.1. *Let $\{x_n, y_n\}$ be a solution of system (6). Then the sequence $\{x_n\}$ is bounded.*

Proof. Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \quad and \quad x_{n_i+1} > x_n, \quad for \quad n < n_i+1. \tag{7}$$

Then, clearly

$$x_{n_i}, y_{n_i} \to 0, \quad y_{n_i-1} \to \frac{\alpha_2}{A_2}, \quad \text{and} \quad x_{n_i-1} \to \infty.$$

In view of Lemma 1.1

$$x_{n_i+1} - x_{n_i-1} = \frac{-y_{n_i}(x_{n_i-1}^2 + y_{n_i-1}x_{n_i-1} - \alpha_1 \cdot \frac{y_{n_i-1}}{y_{n_i}})}{\alpha_1 + y_{n_i}(x_{n_i-1}+y_{n_i-1})} \tag{8}$$

and

$$\frac{y_{n_i-1}}{y_{n_i}} = \frac{y_{n_i-1}}{\alpha_2 + y_{n_i-1}} \cdot (A_2 + x_{n_i-1}),$$

it follows that

$$x_{n_i-1}^2 + y_{n_i-1}x_{n_i-1} - \alpha_1 \cdot \frac{y_{n_i-1}}{y_{n_i}} = x_{n_i-1}^2 + (y_{n_i-1} - \frac{\alpha_1 y_{n_i-1}}{\alpha_2 + y_{n_i-1}})x_{n_i-1} - \frac{\alpha_1 A_2 y_{n_i-1}}{\alpha_2 + y_{n_i-1}} > 0$$

and so

$$x_{n_i+1} < x_{n_i-1}$$

which is a contradiction. ∎

Theorem 3.2. Assume that $A_2 < 1$. *Let $\{x_n, y_n\}$ be a solution of system (6). Assume that $(\bar{x}, \bar{y})$ is the equilibrium point (when it exists) of system (6). Choose initial conditions*

$$x_0 < \gamma_2 - A_2 \ \text{and}\, y_0 > \max\left\{\frac{\alpha_1}{\gamma_2 - A_2}, \bar{y}\right\}.$$

Then

$$\lim_{n\to\infty} x_n = 0 \ \ and \ \lim_{n\to\infty} y_n = \infty.$$

Proof. Indeed,

$$x_1 = \frac{\alpha_1}{x_0 + y_0} < \frac{\alpha_1}{y_0} < 1 - A_2 \quad \text{and} \quad y_1 > \frac{1}{A_2 + x_0} \cdot y_0 > y_0$$

and by induction, we see that

$$x_{n+1} < 1 - A_2 \quad \text{and} \quad y_{n+1} > y_n, \quad \text{for} \quad n \geq 0$$

from which the result follows. ■

4. The Boundedness Character of Solutions of System (12,35)

Here we investigate the boundedness character of solutions of the rational system

$$(12,35): \quad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1}{x_n + y_n} \\ y_{n+1} &= \frac{\beta_2 x_n + \gamma_2 y_n}{1 + x_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{9}$$

with positive parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.

We establish that for every solution $\{x_n, y_n\}$, the first component $\{x_n\}$ is always bounded, for all values of the parameters and for all initial conditions. For the second component, $\{y_n\}$, we show that it is unbounded in a certain region of the parameters and for some initial conditions.

Theorem 4.1. *Let $\{x_n, y_n\}$ be a solution of system* (6). *Then the sequence $\{x_n\}$ is bounded.*

Proof. The proof is along the lines of the proof of Theorem 2.1 and it will be omitted. ■

Theorem 4.2. *Assume that $\gamma_2 > 1$. Let $\{x_n, y_n\}$ be a solution of system* (9). *Assume that $(\bar{x}, \bar{y})$ is the equilibrium point* (*when it exists*) *of system* (9). *Choose initial conditions*

$$x_0 < \gamma_2 - 1 \ \text{ and } y_0 > \max\left\{ \frac{\alpha_1}{\gamma_2 - 1}, \bar{y} \right\}.$$

Then

$$\lim_{n\to\infty} x_n = 0 \ \textit{ and } \lim_{n\to\infty} y_n = \infty.$$

Proof. Indeed,

$$x_1 = \frac{\alpha_1}{x_0 + y_0} < \frac{\alpha_1}{y_0} < \gamma_2 - 1 \quad \text{and} \quad y_1 > \frac{\gamma_2}{1 + x_0} \cdot y_0 > y_0$$

and by induction, we see that

$$x_{n+1} < \gamma_2 - 1 \quad \text{and} \quad y_{n+1} > y_n, \quad \text{for} \quad n \geq 0$$

from which the result follows.

5. The Boundedness Character of System (14,30)

In this section we investigate the boundedness character of solutions of the system:

$$(14,30): \quad \left.\begin{array}{rcl} x_{n+1} & = & \frac{\beta_1 x_n}{1+y_n} \\ \\ y_{n+1} & = & \frac{\alpha_2+\gamma_2 y_n}{B_2 x_n+y_n} \end{array}\right\}, \quad n=0,1,\ldots \tag{10}$$

with positive parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.

We establish that for every solution $\{x_n, y_n\}$, the second component $\{y_n\}$ is always bounded, for all values of the parameters and for all initial conditions. For the first component, $\{x_n\}$, we show that it is unbounded in a certain region of the parameters and for some initial conditions.

Theorem 5.1. *Let $\{x_n, y_n\}$ be a solution of system (10). Then the sequence $\{y_n\}$ is bounded.*

Proof. Assume for the sake of contradiction that there exists a sequence $\{n_i\}$ such that

$$y_{n_i+1} \to \infty.$$

From

$$x_{n+1}y_{n+1} = \frac{\beta_1 x_n}{B_2 x_n + y_n} \cdot \frac{\alpha_2 + \gamma_2 y_n}{1 + y_n}$$

it follows that the sequence $x_{n+1} \cdot y_{n+1}$ is bounded. Then, clearly,

$$x_{n_i}, y_{n_i} \to 0$$

and from this it follows that

$$x_{n_i-1} \to \infty \quad \text{and} \quad y_{n_i-1} \to 0$$

or

$$x_{n_i-1} \to 0 \quad \text{and} \quad y_{n_i-1} \to \infty.$$

For each of the above two cases, and in view of

$$x_{n_i} = \frac{\beta_1 x_{n_i-1}}{1 + y_{n_i-1}} \quad \text{and} \quad y_{n_i} = \frac{\alpha_2 + \gamma_2 y_{n_i-1}}{B_2 x_{n_i-1} + y_{n_i-1}},$$

we obtain a contradiction and the proof is complete. ■

Theorem 5.2. *Assume that $\beta_1 > \gamma_2 + 1$. Let $\{x_n, y_n\}$ be a solution of system (10) with initial conditions (x_0, y_0) such that*

$$x_0 > \max\left\{\frac{\alpha_1}{B_2(\beta_1 - \gamma_2 - 1)}, \bar{x}\right\} \text{ and } y_0 < \beta_1 - 1.$$

Then

$$\lim_{n\to\infty} x_n = \infty \text{ and } \lim_{n\to\infty} y_n = 0.$$

Proof. The proof is along the lines of the proof of Theorem 2.2 and it will be omitted. ■

6. The Boundedness Character of System (29,30)

In this section we investigate the boundedness character of solutions of the system:

$$(29,30):\quad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1+\beta_1 x_n}{1+y_n} \\ y_{n+1} &= \frac{\alpha_2+\gamma_2 y_n}{x_n+y_n} \end{aligned}\right\},\quad n=0,1,\ldots \tag{11}$$

with positive parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.

We establish that for every solution $\{x_n, y_n\}$, the second component $\{y_n\}$ is always bounded, for all values of the parameters and for all initial conditions. For the first component, $\{x_n\}$, we show that it is unbounded in a certain region of the parameters and for some initial conditions.

Theorem 6.1. *Let* $\{x_n, y_n\}$ *be a solution of system* (11). *Then the sequence* $\{y_n\}$ *is bounded*.

Proof. Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$y_{n_i+1} \to \infty.$$

Then, clearly,

$$x_{n_i}, y_{n_i} \to 0.$$

From this and from

$$x_{n_i} = \frac{\alpha_1}{1+y_{n_i-1}} + \frac{\beta_1 x_{n_i-1}}{1+y_{n_i-1}}$$

it follows that

$$y_{n_i-1} \to \infty \quad \text{and} \quad \frac{x_{n_i-1}}{y_{n_i-1}} \to 0.$$

From

$$y_{n_i} = \frac{\alpha_2}{x_{n_i-1}+y_{n_i-1}} + \frac{\gamma_2 \cdot \frac{y_{n_i-1}}{x_{n_i-1}}}{1+\frac{y_{n_i-1}}{x_{n_i-1}}}$$

it follows that

$$y_{n_i} \to \gamma_2$$

which is a contradiction. ■

Theorem 6.2. *Assume that* $\beta_1 > \gamma_2 + 1$. *Let* $\{x_n, y_n\}$ *be a solution of system* (11) *with initial conditions* $(x_0,\ y_0)$ *such that*

$$x_0 > \max\left\{\frac{\alpha_2}{\beta_1-\gamma_2-1}\ ,\ \bar{x}\right\} \text{ and } y_0 < \beta_1 - 1.$$

Then

$$\lim_{n\to\infty} x_n = \infty \text{ and } \lim_{n\to\infty} y_n = 0.$$

Proof. The proof is along the lines of the proof of Theorem 2.2 and it will be omitted. ■

7. The Boundedness Character of System (30,35)

We also investigate in this section the boundedness character of solutions of the system:

$$(30,35):\quad \left.\begin{aligned} x_{n+1} &= \frac{\alpha_1+\beta_1 x_n}{B_1 x_n+y_n} \\ y_{n+1} &= \frac{\beta_2 x_n+\gamma_2 y_n}{A_2+x_n} \end{aligned}\right\},\quad n=0,1,\ldots \tag{12}$$

with positive parameters and with arbitrary nonnegative initial conditions such that the denominators are always positive.
We establish that for every solution $\{x_n,y_n\}$, the first component $\{x_n\}$ is always bounded, for all values of the parameters and for all initial conditions. For the second component, $\{y_n\}$, we show that it is unbounded in a certain region of the parameters and for some initial conditions.

Theorem 7.1. *Let $\{x_n,y_n\}$ be a solution of system* (12). *Then the sequence $\{y_n\}$ is bounded.*

Proof. From

$$x_{n+1}y_{n+1} = \frac{\alpha_1+\beta_1 x_n}{A_1+x_n}\cdot\frac{\beta_2 x_n+\gamma_2 y_n}{B_2 x_n+y_n}$$

it follows that the product $\{x_{n+1}y_{n+1}\}$ is bounded from above and from below.
Assume for the sake of contradiction that there exists a sequence $\{n_i\}$ such that

$$x_{n_i+1}\to\infty.$$

Then, clearly

$$x_{n_i},y_{n_i}\to 0$$

which is a contradiction. ■

Theorem 7.2. *Assume that*

$$\gamma_2 > A_2+\frac{\beta_1}{B_1}.$$

Let $\{x_n,y_n\}$ be a solution of system (12) *with initial conditions* $(x_0,\ y_0)$ *such that*

$$y_0 > \max\left\{\frac{\alpha_1}{\gamma_2-A_2-\frac{\beta_1}{B_1}},\ \bar{y}\right\}\ \textit{and}\ \ x_0<\gamma_2-A_2.$$

Then

$$\lim_{n\to\infty} x_n = 0\ \textit{and}\ \lim_{n\to\infty} y_n=\infty.$$

Proof. The proof is along the lines of the proof of Theorem 2.2 and it will be omitted. ■

8. Conclusions and Future Work

In this paper we investigated the boundedness character of solutions of several systems in the plane. The boundedness character of solutions of a system is one of the main ingredients in understanding the global behavior of a system including global stability. Our future goal is to discover the pattern of boundedness of the complete rational system in the plane

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1+\beta_1 x_n+\gamma_1 y_n}{A_1+B_1 x_n+C_1 y_n} \\ \\ y_{n+1} &= \frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+B_2 x_n+C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \quad . \tag{13}$$

In addition, we want to study systematically the global behavior of all solutions of System (13) and to extend and generalize the results to general systems in two dimensions and higher.

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